A UNIQUENESS RESULT FOR THE DECOMPOSITION OF VECTOR FIELDS IN $\mathbb{R}^d$

STEFANO BIANCHINI AND PAOLO BONICATTO

Dedicated to Alberto Bressan on the occasion of his 60th birthday

Abstract. Given a vector field $\rho(1,\mathbf{b}) \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{R}^{d+1})$ such that $\mathrm{div}_{t,x}(\rho(1,\mathbf{b}))$ is a measure, we consider the problem of uniqueness of the representation η of $\rho(1,\mathbf{b})\mathcal{L}^{d+1}$ as a superposition of characteristics $\gamma : (t^-_\gamma, t^+_\gamma) \to \mathbb{R}^d$, $\dot\gamma(t) = \mathbf{b}(t,\gamma(t))$. We give conditions in terms of a local structure of the representation η on suitable sets in order to prove that there is a partition of $\mathbb{R}^{d+1}$ into disjoint trajectories $\wp_{\mathfrak{a}}$, $\mathfrak{a} \in \mathfrak{A}$, such that the PDE

$$\mathrm{div}_{t,x}\big(u\rho(1,\mathbf{b})\big) \in \mathcal{M}(\mathbb{R}^{d+1}), \qquad u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d),$$

can be disintegrated into a family of ODEs along $\wp_{\mathfrak{a}}$ with measure r.h.s.. The decomposition $\wp_{\mathfrak{a}}$ is essentially unique. We finally show that $\mathbf{b} \in L^1_t(\mathrm{BV}_x)_{\mathrm{loc}}$ satisfies this local structural assumption and this yields, in particular, the renormalization property for nearly incompressible BV vector fields.

Keywords: transport equation, continuity equation, renormalization, uniqueness, Superposition Principle.

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1. Introduction

In this paper we consider the initial value problem for the *continuity equation* associated to a vector field $\mathbf{b} : \mathbb{R}^+_t \times \mathbb{R}^d_x \to \mathbb{R}^d$, i.e.

$$\begin{cases} \partial_t u + \mathrm{div}_x(u\mathbf{b}) = 0, \\ u(0,\cdot) = u_0(\cdot) \end{cases} \tag{1.1}$$

and the corresponding advective formulation, namely the *transport equation*

$$\begin{cases} \partial_t u + \mathbf{b}\cdot\nabla u = 0, \\ u(0,\cdot) = u_0(\cdot) \end{cases} \tag{1.2}$$

where $u\colon \mathbb{R}^+_t \times \mathbb{R}^d_x \to \mathbb{R}$ is a scalar field and $u_0\colon \mathbb{R}^d \to \mathbb{R}$ is a given initial datum. When $\mathbf{b}$ is globally bounded and enjoys Lipschitz bounds, existence and uniqueness results for (classical) solutions to Problems (1.1) and (1.2) are well known. They rely on the so called *method of characteristics* which establishes a deep connection between the "Eulerian" problems (1.1), (1.2) and the "Lagrangian" problem given by the ordinary differential equation driven by $\mathbf{b}$:

$$\begin{cases} \dot\gamma(t) = \mathbf{b}(t,\gamma(t)), & \gamma\colon \mathbb{R}^+ \to \mathbb{R}^d, \\ \gamma(0) = x & x \in \mathbb{R}^d. \end{cases} \tag{1.3}$$

Our aim here is to study the problem of uniqueness in the non-smooth setting. For instance, if we assume that the vector field $\mathbf{b}$ is merely locally integrable, then one can give a distributional meaning to the following equation

$$\mathrm{div}_{t,x}\big(u(1,\mathbf{b})\big) := \partial_t u + \mathrm{div}_x(u\mathbf{b}) = c, \tag{1.4}$$

provided, for instance, $u \in L^\infty(\mathbb{R}^+\times\mathbb{R}^d)$ and $c \in L^1(\mathbb{R}^{d+1})$ is an integrable function. Furthermore, one can prove (see e.g. [DL07]) that, if u is a weak solution of (1.4), then there exists a map $\tilde u \in L^\infty(\mathbb{R}^+\times\mathbb{R}^d)$ such that $u(t,\cdot) = \tilde u(t,\cdot)$ for a.e. $t \in \mathbb{R}^+$ and $t \mapsto \tilde u(t,\cdot)$ is weakly* continuous from $\mathbb{R}^+$ into $L^\infty(\mathbb{R}^d)$ and

this allows to prescribe an initial condition for a weak solution u of (1.4), by imposing that $u(0,\cdot) = u_0(\cdot)$ holds if $\widetilde{u}(0,\cdot) = u_0(\cdot)$.
The definition of weak solutions to the transport equation (1.2) is slightly more delicate: if the spatial distributional divergence of $\mathbf{b}$ is a measure which is absolutely continuous with respect to the Lebesgue measure, then the equation in (1.2) can be written in the form (1.4) and, as already pointed out, the latter can be understood in the sense of distributions.
In the case when $\mathrm{div}_x \mathbf{b}$ is a measure which has a non trivial singular part, the notion of weak solution of (1.2) can be defined within the class of *nearly incompressible* vector fields.

Definition 1.1. A locally integrable vector field $\mathbf{b}\colon \mathbb{R}^+_t \times \mathbb{R}^d_x \to \mathbb{R}^d$ is called *nearly incompressible* if there exists a function $\rho\colon \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ (called *density* of $\mathbf{b}$) and a constant $C > 0$ such that $C^{-1} \le \rho(t,x) \le C$ for Lebsgue almost every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$ and

$$\mathrm{div}_{t,x}\big(\rho(1,\mathbf{b})\big) = 0 \qquad \text{in the sense of distributions on } \mathbb{R}^+ \times \mathbb{R}^d.$$

Accordingly, one can give the following definition of weak solution:

Definition 1.2. Let $\mathbf{b}$ be a nearly incompressible vector field with density ρ. We say that a function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ is a ρ-weak solution of (1.2) if

$$\mathrm{div}_{t,x}\big(\rho u(1,\mathbf{b})\big) = 0 \qquad \text{in the sense of distributions on } \mathbb{R}^+ \times \mathbb{R}^d.$$

Thanks to Definition 1.2 one can prescribe the initial condition for a weak solution to the transport equation similarly to the case of the continuity equation, which we mentioned above (see [DL07] for the details).

1.1. **The classical approach: renormalized solutions.** Once reasonable definitions of weak solutions have been established, one can start wondering whether they exist and are unique.
On the one hand, *existence* results for weak solutions are available under quite mild assumptions on $\mathbf{b}$, due to the linearity of the problems: for instance, one can show that a weak solution to initial value problem for transport equation (1.2) with a nearly incompressible vector field always exists by means of a standard regularization argument (see [DL07]).
On the other hand, the problem of *uniqueness* of weak solutions is more delicate and has been studied by several authors, since the work of DiPerna-Lions [DL89]. In that paper, uniqueness was established as a corollary of the so called *renormalization property*. Roughly speaking, a bounded function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ is said to be a *renormalized solution* to (1.2) if for all $\beta \in C^1(\mathbb{R})$ the function $\beta(u)$ is a solution to the corresponding Cauchy problem:

$$\begin{cases} \partial_t u + \mathbf{b}\cdot\nabla u = 0, \\ u(0,\cdot) = u_0 \end{cases} \implies \begin{cases} \partial_t(\beta(u)) + \mathbf{b}\cdot\nabla(\beta(u)) = 0 \\ \beta(u(0,\cdot)) = \beta(u_0(\cdot)) \end{cases} \qquad \text{for every } \beta \in C^1(\mathbb{R}).$$

This can be interpreted as a sort of weak "Chain Rule" for the function u, saying that u is differentiable along the flow generated by $\mathbf{b}$. In [DL89] it is shown that the validity of this property for every $\beta \in C^1(\mathbb{R})$ implies, under general assumptions, uniqueness of weak solutions for (1.2); furthermore, it is proved that renormalization property is fulfilled by vector fields $\mathbf{b}$ which have locally Sobolev regularity (in space).
The argument relies on an approximation scheme based on *commutator estimates*: if $\{\varphi^\varepsilon\}_{\varepsilon>0}$ is a standard family of mollifiers in $\mathbb{R}^n$ and $u^\varepsilon := u * \varphi^\varepsilon$ then one can write

$$\partial_t u^\varepsilon + \mathbf{b}\cdot\nabla u^\varepsilon = T^\varepsilon \tag{1.5}$$

where T^ε is the *commutator* defined as

$$T^\varepsilon := \mathbf{b}\cdot\nabla u^\varepsilon - (\mathbf{b}\cdot\nabla u) * \varphi^\varepsilon$$

By multiplying both sided of (1.5) times $\beta'(u^\varepsilon)$ one obtains

$$\partial_t \beta(u^\varepsilon) + \mathbf{b}\cdot\nabla(\beta(u^\varepsilon)) = T^\varepsilon \beta'(u^\varepsilon).$$

If $\mathbf{b} \in W^{1,p}_x$, one can show that the r.h.s. converges *strongly* (in L^p) to 0, from which one deduces the renormalization property.

1.2. **Bressan's compactness conjecture.** In the recent years, several efforts have been made in order to extend these results to a larger class of vector fields, remarkably BV vector fields. In 2004, Ambrosio [Amb04] proved the renormalization property for vector fields of bounded variation whose divergence is absolutely continuous.
However, in view of the relevant connections with the theory of hyperbolic systems of conservation laws (for instance the Keyfitz-Kranzer system, see [KK80]), it is interesting to consider the case of vector fields whose divergence has non trivial singular part. In particular, one would like to extend the DiPerna-Lions theory of renormalized solutions to the transport equation when the vector field $\mathbf{b}$ is BV (locally in space) and nearly incompressible. Notice that, using simple regularization arguments, one can prove that if $\operatorname{div} \mathbf{b} \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ then $\mathbf{b}$ is nearly incompressible. The converse implication does not hold, so near incompressibility can be considered as a weaker version of the assumption $\operatorname{div} \mathbf{b} \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$. In particular, uniqueness of solutions in the BV nearly incompressible setting implies (as shown in [ADLM07]) the following conjecture on compactness of ODEs, raised by A. Bressan in 2003 (see [Bre03a, Bre03b]:

Conjecture (Bressan's compactness conjecture)**.** *Let $\mathbf{b}_n \colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $n \in \mathbb{N}$, be a sequence of smooth vector fields. Denote by Φ_n the flow generated by $\mathbf{b}_n$, i.e.*

$$\begin{aligned} \frac{d}{dt}\Phi_n(t,x) &= \mathbf{b}_n(t, \Phi_n(t,x)), \\ \Phi_n(0,x) &= x. \end{aligned}$$

Assume that $\|\mathbf{b}_n\|_\infty + \|\nabla_{t,x}\mathbf{b}_n\|_{L^1}$ is uniformly bounded and there exists a constant $C > 0$ such that

$$C^{-1} \le \det(\nabla_x \Phi_n(t,x)) \le C$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ and all $n \in \mathbb{N}$. Then the sequence Φ_n is strongly precompact in L^1_{loc}.

1.2.1. *The chain rule approach and the 2D case.* In view of Definition 1.2, in order to establish uniqueness of solutions to transport equation (1.2) in the nearly incompressible case (and, thus, Bressan's conjecture), one is led to consider continuity equation

$$\partial_t \rho + \operatorname{div}_x(\rho \mathbf{b}) = 0 \tag{1.6}$$

and would like to prove a uniqueness result for the initial value problem associated to (1.6). By pursuing the renormalization approach, one has to observe that the chain rule along the flow of $\mathbf{b}$ for the equation (1.6) takes a different form (even in the smooth setting):

$$\partial_t \beta(\rho) + \operatorname{div}_x\big(\beta(\rho)\mathbf{b}\big) = \big(\beta(\rho) - \rho\beta'(\rho)\big)\operatorname{div}_x \mathbf{b}. \tag{1.7}$$

In general, the r.h.s. of (1.7) cannot be written in that form, being only a distribution; in the case the vector field $\mathbf{b} \in \mathrm{BV}$, it can be shown to be a measure, controlled by $\operatorname{div}_x \mathbf{b}$. As noted in [ADLM07], the main problem is to give a meaning to the r.h.s. of (1.7) when the measure $\operatorname{div}_x \mathbf{b}$ is singular and ρ is only defined almost everywhere with respect to Lebesgue measure. To overcome this difficulty, the authors split $\operatorname{div}_x \mathbf{b} \in \mathcal{M}(\mathbb{R}^{d+1})$ into its absolutely continuous part, jump part and Cantor part and treat the cases separately. Their first result ([ADLM07, Thm. 3]) is that in all Lebesgue points of ρ the formula (1.7) holds (possibly being $\operatorname{div}_x \mathbf{b}$ singular), where ρ is replaced by its Lebsgue value $\tilde\rho$. This is achieved along the same techniques of [Amb04], which are in turn a (non-trivial) extension of the ones employed in [DL89]: in general, in the BV setting, strong convergence of commutators does not hold. The argument can be fixed if $\operatorname{div}_x \mathbf{b}$ is absolutely continuous, by considering suitable convolution kernels which look more elongated in some directions in order to control the singular terms.
By exploiting properties of Anzellotti's weak *normal traces* for measure divergence vector fields (see [Anz83]), Ambrosio-De Lellis-Malý managed to settle the jump part: they obtain an explicit formula (in the spirit (1.7)), involving the traces of $\mathbf{b}$ and $\rho(1, \mathbf{b})$ along a $\mathcal{H}^{d-1}$-rectifiable set. We refer the reader also to [ACM05] for an extension of these results to the BD case.
In order to tackle the Cantor part, a "transversality condition" between the vector field and its derivative is assumed: it is shown that, if in a point $(\bar t, \bar x)$ one has $(D\mathbf{b} \cdot \mathbf{b})(\bar t, \bar x) \neq 0$ (where $\mathbf{b}(\bar t, \bar x)$ is the Lebesgue value of $\mathbf{b}$ in $(\bar t, \bar x)$) then the point $(\bar t, \bar x)$ is a Lebesgue point for ρ.
From the analysis of [ADLM07], it remains open the case of tangential points, i.e. the set of points at which $D\mathbf{b} \cdot \mathbf{b}$ vanishes, which make up the so called *tangential set*. This is actually relevant, as shown in [BG16]: answering negatively to one of the questions in [ADLM07], in [BG16] the authors exhibited

Figure 1. Example of [BG16]: the tangential set of the vector field $\mathbf{b}$ (only the integral curves have been drawn here) is a Cantor like set of dimension $3/2$. Notice that each trajectory γ meets the tangential set in exactly one point, at time t_γ: the density ρ, computed along the curve, is piecewise constant, having a unique jump of size 1 in t_γ.

an example of BV, nearly incompressible vector field with non empty tangential set. Even worse, the tangential set is a Cantor like set of non integer dimension but, at level of the density ρ, one sees a pure jump. This severe pathology is depicted in Figure 1 and we refer the reader to [BG16] for a detailed construction.

In the same paper, the authors studied the 2D case and established, via a different technique, uniqueness of weak solutions for a BV vector field, nearly incompressible (with a time independent density): as a consequence they give explicitly the form of the r.h.s. of (1.7) in the 2D case. Their proof is inspired to previous results in the divergence free case (obtained by [ABC14]) and takes advantage of the underlying Hamiltonian structure of the problem in the two dimensional setting. In [ABC14], the existence of a Lipschitz Hamiltonian allows to establish a sufficient and necessary condition for uniqueness for autonomous, divergence-free, bounded vector fields: this is done via a disintegration argument, in view of the regularity results for level sets of Lipschitz maps obtained in [ABC13]. For an extension of these techniques to the 2D nearly incompressible case we refer the reader to [BBG16].

1.3. **A different method.** Our analysis starts from the following observation: the two techniques presented above (Hamiltonian in two-dimensional setting and Chain Rule) are not suited for the general case for two different reasons. On the one hand, in the general d-dimensional case with $d > 2$, the Hamiltonian approach cannot be applied, as divergence free vector fields in $\mathbb{R}^d$ do not admit in general a Lipschitz potential. On the other hand, in the Chain Rule approach the problem is more subtle: clearly, it seems arduous to construct suitable convolution kernels, which adapt to the irregularity of the vector field, controlling the errors, once the main term is exhibited. The subtle problem is however to determine *which* are the main terms: one has to compute some sort of trace on sets which are not rectifiable, i.e. Cantor-like sets. Lacking a suitable notion of trace, this task seems quite difficult. Such a notion could be given by means of a *Lagrangian representation* η of the $\mathbb{R}^{d+1}$-valued vector field $\rho(1, \mathbf{b})$, and this is the starting point of our approach.

Lagrangian representations. In the general non-smooth setting, one could recover a link between the continuity equation (1.1) and the ODE (1.3) thanks to the so called *Superposition Principle*, which has been established by Ambrosio in [Amb04] (see also [Smi94]). Roughly speaking, it asserts that, if the vector field is globally bounded, every non-negative (possibly measure-valued) solution to the PDE (1.1) can be written as a superposition of solutions obtained via propagation along integral curves of $\mathbf{b}$, i.e. solutions to the ODE (1.3).

More generally, let us consider a locally integrable vector field $\mathbf{b} \in L^1_{\mathrm{loc}}((0,T) \times \mathbb{R}^d)$ and let ρ be a non-negative solution to the balance law

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{b}) = \mu, \qquad \mu \in \mathcal{M}((0,T) \times \mathbb{R}^d). \tag{1.8}$$

with $\rho \in L^1_{\mathrm{loc}}((1+|\mathbf{b}|)\mathcal{L}^{d+1})$ (so that a distributional meaning can be given). For simplicity, we will often write (1.8) in the shorter form

$$\operatorname{div}_{t,x}\big(\rho(1,\mathbf{b})\big) = \mu. \tag{1.9}$$

Let us denote the space of continuous curves by

$$\varUpsilon := \big\{(t_1, t_2, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ \times C(\mathbb{R}^+, \mathbb{R}^d),\, t_1 < t_2\big\}$$

and let us tacitly identify the triplet $(t^-_\gamma, t^+_\gamma, \gamma) \in \varUpsilon$ with γ, so that we will simply write $\gamma \in \varGamma$. We say that a finite, non negative measure η over the set $\varUpsilon$ is a *Lagrangian representation* of the vector field $\rho(1, \mathbf{b})$ if the following conditions hold:

(1) η is concentrated on the set of characteristics $\varGamma$, defined as

$$\varGamma := \{(t_1, t_2, \gamma) \in \varUpsilon : \gamma \text{ characteristic of } \mathbf{b} \text{ in } (t_1, t_2)\}\,;$$

we explicitly recall that a curve γ is said to be a characteristic of the vector field $\mathbf{b}$ in the interval I_γ if it is an absolutely continuous solutions to the ODE

$$\dot\gamma(t) = \mathbf{b}(t, \gamma(t)),$$

in I_γ, which means that for every $(s,t) \subset I_\gamma$ we have

$$\int_\Gamma \left| \gamma(t) - \gamma(s) - \int_s^t \mathbf{b}(\tau, \gamma(\tau))\, d\tau \right| \eta(d\gamma) = 0.$$

(2) The solution ρ can be seen as a superposition of the curves selected by η, i.e. if $(\mathrm{id}, \gamma) \colon I_\gamma \to I_\gamma \times \mathbb{R}^d$ denotes the map defined by $t \mapsto (t, \gamma(t))$, we ask that

$$\rho \mathcal{L}^{d+1} = \int_\Gamma (\mathrm{id}, \gamma)_\sharp \mathcal{L}^1 \,\eta(d\gamma);$$

(3) we can decompose μ, the divergence of $\rho(1, \mathbf{b})$, as a local superposition of Dirac masses without cancellation, i.e.

$$\begin{aligned} \mu &= \int_\Gamma \Big[\delta_{t^-_\gamma, \gamma(t^-_\gamma)}(dt\, dx) - \delta_{t^+_\gamma, \gamma(t^+_\gamma)}(dt\, dx) \Big] \eta(d\gamma), \\ |\mu| &= \int_\Gamma \Big[\delta_{t^-_\gamma, \gamma(t^-_\gamma)}(dt\, dx) + \delta_{t^+_\gamma, \gamma(t^+_\gamma)}(dt\, dx) \Big] \eta(d\gamma). \end{aligned}$$

The existence of such a decomposition into curves is a consequence of general structural results of 1-dimensional normal currents (see [Smi94] and, for the case $\mu = 0$, [AC08, Thm. 12]). The non-negativity assumption on $\rho \geq 0$ (i.e. the *a-cyclicity* of $\rho(1, \mathbf{b})$ in the language of currents) plays here a role, allowing to reparametrize the curves in such a way they become characteristic of $\mathbf{b}$, i.e. they satisfy Point (1).

Restriction of Lagrangian representations and proper sets. One problem we face immediately lies in the fact that η is a *global* object, thus it is not immediate to relate suitable *local estimates* with η: in other words, in general, η cannot be restricted to a set, without losing the property of being a Lagrangian representation. If we are given an open set $\Omega \subset \mathbb{R}^{d+1}$ and a curve γ, we can write

$$\gamma^{-1}(\Omega) = \bigcup_{i=1}^\infty (t^{i,-}_\gamma, t^{i,+}_\gamma)$$

and then consider the family of curves

$$\mathtt{R}^i_\Omega \gamma := \gamma \llcorner_{(t^{i,-}_\gamma, t^{i,+}_\gamma)}.$$

We can now define

$$\eta_\Omega := \sum_{i=1}^\infty (\mathtt{R}^i_\Omega)_\sharp \eta. \tag{1.10}$$

In general, the series in (1.10) does not converge. Moreover, even if the quantity in (1.10) is well defined as a measure, since η satisfies Points (1) and (2) of the definition of Lagrangian representation 3.1, it certainly holds

$$\rho(1, \mathbf{b})\, \mathcal{L}^{d+1} \llcorner_\Omega = \int_\Gamma (\mathrm{id}, \gamma)_\sharp \big((1, \dot\gamma) \mathcal{L}^1 \big)\, \eta_\Omega(d\gamma).$$

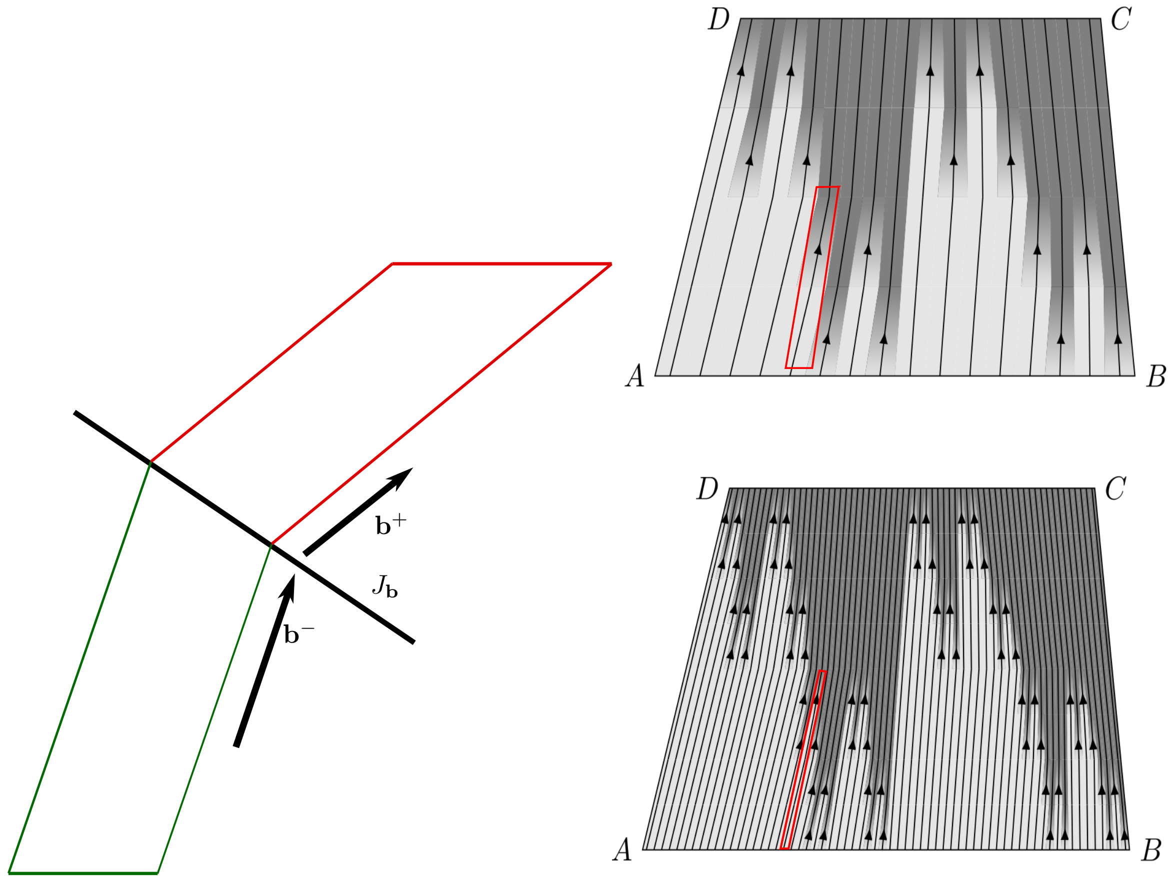
but, in general, Point (3) is not satisfied by η_Ω (more precisely the second formula): in other words, η_Ω might not be a Lagrangian representation of $\rho(1, \mathbf{b}) \mathcal{L}^{d+1} \llcorner_\Omega$: the key point is that the sets of γ which are exiting from or entering in Ω are not disjoint.

Thus the first question we have to answer to is to characterize the open sets $\Omega \subset \mathbb{R}^{d+1}$ for which η_Ω is a Lagrangian representation of $\rho(1, \mathbf{b}) \mathcal{L}^{d+1} \llcorner_\Omega$. It turns out that there are sufficiently many open sets Ω with this property: apart from having a piecewise C^1-regular boundary and assuming that $\mathcal{H}^d \llcorner_{\partial\Omega}$-a.e. point is a Lebesgue point for $\rho(1, \mathbf{b})$, the fundamental fact is that there are two Lipschitz functions $\phi^{\delta,\pm}$ such that

$$\mathbb{1}_\Omega \leq \phi^{\delta,+} \leq \mathbb{1}_{\Omega + B^{d+1}_\delta(0)}, \qquad \mathbb{1}_{\mathbb{R}^{d+1} \setminus \Omega} \leq \phi^{\delta,-} \leq \mathbb{1}_{\mathbb{R}^{d+1} \setminus \Omega + B^{d+1}_\delta(0)}$$

and

$$\lim_{\delta \to 0} \rho |(1, \mathbf{b}) \cdot \nabla \phi^{\delta,\pm}|\, \mathcal{L}^{d+1} = \rho |(1, \mathbf{b}) \cdot \mathbf{n}|\, \mathcal{H}^d \llcorner_{\partial\Omega} \qquad \text{in the sense of measures on } \mathbb{R}^{d+1},$$



(a) Visual proof of Thm. 6.2 in [ACM05]: the trace of ρ on the jump set $J_{\mathbf{b}}$ is strong, as a consequence of the fact that the flux of $\mathbf{b}$ through the lateral boundaries of the cylinders is asymptotically negligible w.r.t. the base of the cylinders.

(b) Back to the example of [BG16]: in the tangential points, the cylinders should be much more elongated than in the jump case, as the vector field is not transversal.

Figure 2. Strong traces via cylinders: the jump case and the Cantor case.

which essentially mean that $\rho(1,\mathbf{b})\mathcal{H}^d \llcorner_{\partial\Omega}$ is measuring the flux of $\rho(1,\mathbf{b})$ across $\partial\Omega$. We call these set $\rho(1,\mathbf{b})$-*proper* (or just *proper* for shortness): our first results are that there are sufficiently many proper sets and that they can be perturbed in order to adapt to the vector field under study.

Cylinders of approximate flow. Once we are able to localize the problem in a proper set, we can start studying which are the pieces of information on the local behavior of the vector field that one needs in order to deduce global uniqueness results. To begin with, we consider again the case of the jump part of $\mathbf{b}$ in the $L^\infty \cap \mathrm{BV}$ (or $L^\infty \cap \mathrm{BD}$) case: in this framework, in [ACM05, Thm. 6.2] it has been proved the existence of a *strong* trace for ρ over the jump set of $\mathbf{b}$ by taking suitable cylinders, so that on both sides of the discontinuity the later flux becomes negligible w.r.t. their base (see Figure 2a). By *strong trace* we mean that the trace operator, defined in the Anzellotti's distributional sense, agrees with the (approximate) pointwise limits defined with integral averages on balls. One could be tempted at this point to reproduce the proof in the tangential points: ignoring the fact that we do not have a suitable notion of (strong) trace on these Cantor sets, the main difference lies in the fact that, since the vector field is not transversal to the measure theoretic normal of the set, the cylinders should be much more elongated (see Figure 2b).

Thus we have to look for a different approach. Given a proper set $\Omega \subset \mathbb{R}^{d+1}$, we assume we can construct locally *cylinders of approximate flow* as follows:

Assumption 1.3. There are constants $\mathtt{M}, \varpi > 0$ and a family of functions $\{\phi^\ell_\gamma\}_{\ell>0,\gamma\in\Gamma}$ such that:

(1) for every $\gamma \in \Gamma, \ell \in \mathbb{R}^+$, the function $\phi^\ell_\gamma \colon [t^-_\gamma, t^+_\gamma] \times \mathbb{R}^d \to [0,1]$ is Lipschitz, so that it can be used as a test function;
(2) the *shrinking ratio* of the cylinder ϕ^ℓ_γ is controlled in time, preventing it collapses to a point: more precisely, for $t \in [t^-_\gamma, t^+_\gamma]$ and $x \in \mathbb{R}^d$,

$$\mathbb{1}_{\gamma(t)+B^d_{\ell/\mathtt{M}}(0)}(x) \leq \phi^\ell_\gamma(t,x) \leq \mathbb{1}_{\gamma(t)+B^d_{\mathtt{M}\ell}(0)}(x);$$

(3) we control in a quantitative way the flux through the "lateral boundary of the cylinder" (compared to the total amount of curves starting from the "base of the cylinder") with the quantity ϖ: more precisely, denoting by

$$\begin{aligned}\mathsf{Flux}^\ell(\gamma) &:= \begin{array}{c}\text{flux of the the vector field } \rho(1,\mathbf{b})\\ \text{across the "boundary of the cylinder" } \phi^\ell_\gamma\end{array}\\ &= \iint_{(t^-_\gamma,t^+_\gamma)\times\mathbb{R}^d} \rho(t,x)\big|(1,\mathbf{b})\cdot\nabla\phi^\ell_\gamma(t,x)\big|\,\mathcal{L}^{d+1}(dx\,dt),\end{aligned}$$

$$\sigma^\ell(\gamma) := \text{amount of curves starting from the base of the cylinder } \phi^\ell_\gamma$$

and

$$\eta^{\mathrm{in}}_\Omega := \eta_{\Omega\llcorner\{\text{curves entering in }\Omega\}}$$

we ask that

$$\int_\Gamma \frac{1}{\sigma^\ell(\gamma)}\mathsf{Flux}^\ell(\gamma)\,\eta^{\mathrm{in}}_\Omega(d\gamma) \leq \varpi. \tag{1.11}$$

We decided to call *cylinders of approximate flow* the family of functions $\{\phi^\ell_\gamma\}_{\ell>0,\gamma\in\Gamma}$: indeed, if γ is a characteristic of the vector field $\mathbf{b}$, the function ϕ^ℓ_γ can be thought as generalized, smoothed cylinder centered at γ. Notice that the measure $\eta^{\mathrm{in}}_\Omega$ makes sense if Ω is a proper set, in view of the above analysis. Thus the ultimate meaning of the assumption is that one controls the ratio between the flux of $\rho(1,\mathbf{b})$ across the lateral boundary of the cylinders and the total amount of curves entering through its base in a uniform way (w.r.t. ℓ), as the cylinder shrinks to a trajectory γ. A completely similar computation can be performed backward in time, by considering η_Ω restricted to the exiting trajectories and adopting suitable modifications.

Passing to the limit via transport plans. At this point, one would like to determine what the cylinder estimate (1.11) yields in the limit $\ell \to 0$. In order to perform this passage to the limit, we borrow some tools from the Optimal Transportation Theory. The language of transference plans is particularly suited for our purposes: we define

$$\Gamma^{\mathrm{cr}}(\Omega) := \big\{\gamma \in \Gamma : \gamma(t^\pm_\gamma) \in \partial\Omega\big\}, \qquad \Gamma^{\mathrm{in}}(\Omega) := \big\{\gamma \in \Gamma : \gamma(t^-_\gamma) \in \partial\Omega\big\}$$

and we consider plans between $\eta^{\mathrm{cr}}_\Omega := \eta_{\Omega\llcorner\Gamma^{\mathrm{cr}}(\Omega)}$ and the entering trajectory measure $\eta^{\mathrm{in}}_\Omega$. Notice that $\eta^{\mathrm{cr}}_\Omega$ is concentrated, by definition, on the set of trajectories entering in and exiting from Ω (*crossing* trajectories). In the correct estimate one has to take into account also of trajectories which end inside the set Ω and this, in view of Point 3 of the definition of Lagrangian representation, is estimated by the negative part μ^- of the divergence μ, defined in (1.9). By means of a deep duality result in Optimal Transport [Kel84], one obtains the following

Proposition 1.4. *Let $\Omega \subset \mathbb{R}^{d+1}$ be a proper set and η be a Lagrangian representation of $\rho(1,\mathbf{b})$. If Assumption 1.3 holds then there exist $N_1 \subset \Gamma^{\mathrm{cr}}(\Omega), N_2 \subset \Gamma^{\mathrm{in}}(\Omega)$ such that*

$$\eta^{\mathrm{cr}}_\Omega(N_1) + \eta^{\mathrm{in}}_\Omega(N_2) \leq \inf_{C>1}\left\{2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1}\right\}$$

and for every $(\gamma,\gamma') \in (\Gamma^{\mathrm{cr}} \setminus N_1) \times (\Gamma^{\mathrm{in}} \setminus N_2)$

$$\textit{either } \operatorname{clos}\operatorname{Graph}\gamma' \subset \operatorname{clos}\operatorname{Graph}\gamma \textit{ or } \operatorname{clos}\operatorname{Graph}\gamma, \operatorname{clos}\operatorname{Graph}\gamma' \textit{ are disjoint.} \tag{$\star$}$$

Proposition 1.4 gives essentially a uniqueness result (from the Lagrangian point of view) at a *local* level, namely inside a proper set Ω: it says that, under Assumption 1.3, up to removing a set of trajectories whose measure is controlled, one gets a family of essentially disjoint trajectories (meaning that are either disjoint or one contained in the other).

***Untangling* of trajectories.** It seems at this point natural to try to perform some "local-to-global" argument, seeking a global analog of Proposition 1.4. In order to do this, we introduce the following *untangling functional for* η^{in}, defined on the class of proper sets as

$$\mathfrak{f}^{\mathrm{in}}(\Omega) := \inf\Big\{\eta^{\mathrm{cr}}_{\Omega}(N_1) + \eta^{\mathrm{in}}_{\Omega}(N_2) : \forall(\gamma,\gamma') \in (\mathit{\Gamma}\setminus N_1)\times(\mathit{\Gamma}\setminus N_2) \text{ the condition } (\star) \text{ holds }\Big\}$$

and, in a similar fashion, one can define an untangling functional for the trajectories that are exiting from the domain Ω. In a sense, these functionals are measuring the minimum amount of curves one has to remove so that the remaning ones are essentially disjoint, i.e. they satisfy condition $(\star)$. The main property of these functionals is that they are subadditive with respect to the domain Ω, meaning that

$$\mathfrak{f}^{\mathrm{in}}(\Omega) \leq \mathfrak{f}^{\mathrm{in}}(\mathrm{U}) + \mathfrak{f}^{\mathrm{in}}(\mathrm{V}),$$

whenever $\mathrm{U},\mathrm{V}\subset\mathbb{R}^{d+1}$ are proper sets whose union $\Omega := \mathrm{U}\cup\mathrm{V}$ is proper. The subadditivity suggests the possibility of having a local control in terms of a measure ϖ^τ, whose mass is $\tau > 0$, replacing the constant ϖ in Proposition 1.4 with $\varpi^\tau(\Omega)$. In view of Proposition 1.4 one has to combine ϖ^τ with the divergence and this can be done by introducing a suitable measure $\zeta^\tau_C \approx C\varpi^\tau + \frac{|\mu|}{C}$ on $\mathbb{R}^{d+1}$. If Assumption 1.3 is satisfied locally by a suitable family of balls, then one can show, by means of a covering argument, the following fundamental proposition, which is the global analog of Proposition 1.4.

Proposition 1.5. *There exists a set of trajectories $N \subset \mathit{\Gamma}$ such that*

$$\eta(N) \leq C_d \zeta^\tau_C(\mathbb{R}^{d+1})$$

and for every $(\gamma,\gamma')\in(\mathit{\Gamma}\setminus N)^2$ it holds

$$\begin{aligned} &\textit{either } \operatorname{Graph}\gamma \subset \operatorname{Graph}\gamma' \textit{ or } \operatorname{Graph}\gamma' \subset \operatorname{Graph}\gamma \\ &\textit{or } \operatorname{Graph}\gamma, \operatorname{Graph}\gamma' \textit{ are disjoint (up to the end points).} \end{aligned} \tag{$\star\star$}$$

The interesting situation is when the measure ζ^C_τ can be taken arbitrarily small, i.e. when $\tau \to 0$: in that case η is said to be *untangled*, i.e. it is concentrated on a set $\mathit{\Delta}$ such that for every $(\gamma,\gamma')\in\mathit{\Delta}\times\mathit{\Delta}$ the condition $(\star\star)$ holds.

Partition via characteristics and Lagrangian uniqueness. The *untangling* of trajectories is the core of our approach and it encodes, in our language, the uniqueness issues and the computation of the chain rule. Indeed, once the untangled set $\mathit{\Delta}$ is selected, we can construct an equivalence relation on it, identifying trajectories whenever they coincide in some time interval: this gives a partition of $\mathit{\Delta}$ into equivalence classes $E_{\mathfrak{a}} := \{\wp_{\mathfrak{a}}\}_{\mathfrak{a}}$, being $\mathfrak{A}$ a suitable set of indexes. This, in turn, induces a partition of $\mathbb{R}^{d+1}$ (up to a set $\rho\mathcal{L}^{d+1}$-negligible) into disjoint trajectories (that we still denote by $\wp_{\mathfrak{a}}$): both partitions admit a Borel section (i.e. there exist Borel functions $\mathtt{f}\colon\mathbb{R}^{d+1}\to\mathfrak{A}$ and $\hat{\mathtt{f}}\colon\mathit{\Delta}\to\mathfrak{A}$ such that $\wp_{\mathfrak{a}} = \mathtt{f}^{-1}(\mathfrak{a})$ and $\hat{\mathtt{f}}^{-1}(\mathfrak{a}) = E_{\mathfrak{a}}$ for every $\mathfrak{a}\in\mathfrak{A}$): hence a disintegration approach can be adopted, like in the two-dimensional setting. One reduces the PDE (1.9) into a family of one-dimensional ODEs along the trajectories $\{\wp_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathfrak{A}}$: we are thus recovering a sort of method of the characteristic in the weak setting.

To formalize this disintegration issue, we propose to call a Borel map $\mathtt{g}\colon\mathbb{R}^{d+1}\to\mathfrak{A}$ a *partition via characteristics* of the vector field $\rho(1,\mathbf{b})$ if:

- for every $\mathfrak{a}\in\mathfrak{A}$, $\mathtt{g}^{-1}(\mathfrak{a})$ coincides with $\operatorname{Graph}\gamma_{\mathfrak{a}}$, where $\gamma_{\mathfrak{a}}\colon I_{\mathfrak{a}}\to\mathbb{R}^{d+1}$ is a characteristic of $\mathbf{b}$ in some open domain $I_{\mathfrak{a}}\subset\mathbb{R}$;
- if $\hat{\mathtt{g}}$ denotes the corresponding map $\hat{\mathtt{g}}\colon\mathit{\Delta}\to\mathfrak{A}$, $\hat{\mathtt{g}}(\gamma) := \mathtt{g}(\operatorname{Graph}\gamma)$, setting $m := \hat{\mathtt{g}}_\sharp\eta$ and letting $w_{\mathfrak{a}}$ be the disintegration

$$\rho\,\mathcal{L}^{d+1} = \int_{\mathfrak{A}} (\mathrm{id},\gamma_{\mathfrak{a}})_\sharp(w_{\mathfrak{a}}\mathcal{L}^1)\,m(d\mathfrak{a})$$

then

$$\frac{d}{dt}w_{\mathfrak{a}} = \mu_{\mathfrak{a}} \in \mathcal{M}(\mathbb{R}), \tag{1.12}$$

where $w_{\mathfrak{a}}$ is considered extended to 0 outside the domain of $\gamma_{\mathfrak{a}}$;
- it holds
$$\mu = \int (\mathrm{id}, \gamma_{\mathfrak{a}})_\sharp \mu_{\mathfrak{a}} \, m(d\mathfrak{a}) \qquad \text{and} \qquad |\mu| = \int (\mathrm{id}, \gamma_{\mathfrak{a}})_\sharp |\mu_{\mathfrak{a}}| \, m(d\mathfrak{a}).$$

We will say the partition is *minimal* if moreover
$$\lim_{t \to \bar t \pm} w_{\mathfrak{a}}(t) > 0 \qquad \forall \bar t \in I_{\mathfrak{a}}.$$

In view of the discussion above, the family of equivalence classes $\{\wp_{\mathfrak{a}}\}_{\mathfrak{a} \in \mathfrak{A}}$ arising from the untangled set Δ constitutes a partition via characteristics. Moreover, since the function $w_{\mathfrak{a}}$ is a BV function on $\mathbb{R}$, in view of (1.12), we can further split the equivalence classes so that it becomes a minimal partition via characteristics of $\rho(1, \mathbf{b})$. We have thus obtained the following

Main Theorem 1. *There exists a minimal partition via characteristics* $\mathtt{f}$ *of* $\rho(1, \mathbf{b})\mathcal{L}^{d+1}$.

If now $u \in L^\infty$ is such that $\mathrm{div}_{t,x}(u\rho(1, \mathbf{b})) = \mu'$ is a measure, one can repeat the computations for $(2\|u\|_\infty + u)\rho(1, \mathbf{b})\mathcal{L}^{d+1}$ obtaining that the same partition via characteristics works also for $u\rho(1, \mathbf{b})\mathcal{L}^{d+1}$, concluding with the following uniqueness result, which is the second main result of the present work.

Main Theorem 2. *If* $u \in L^\infty(\rho\mathcal{L}^{d+1})$ *then the map* $\mathtt{f}$ *is a partition via characteristics of* $u\rho(1, \mathbf{b})\mathcal{L}^{d+1}$.

In particular, by disintegrating the PDE $\mathrm{div}(u\rho(1, \mathbf{b})) = \mu'$ along the characteristics $\wp_{\mathfrak{a}} = \mathtt{f}^{-1}(\mathfrak{a})$, we obtain the one-dimensional equation
$$\frac{d}{dt}\Big(u\big(t, \wp_{\mathfrak{a}}(t)\big) w_{\mathfrak{a}}(t) \Big) = \mu'_{\mathfrak{a}}.$$

At this point, an application of Volpert's formula for one-dimensional BV functions allows an explicit computation of $\frac{d}{dt}(\beta(u \circ \wp_a) w_{\mathfrak{a}})$, i.e. of $\mathrm{div}(\beta(u)\rho(1, \mathbf{b}))$ thus establishing the Chain rule in the general setting. We remark that, even without BV-BD bounds on $\mathbf{b}$, the distribution $\mathrm{div}(\beta(u)\rho(1, \mathbf{b}))$ turns out to be a measure in our setting, i.e. when the representation η is untangled.

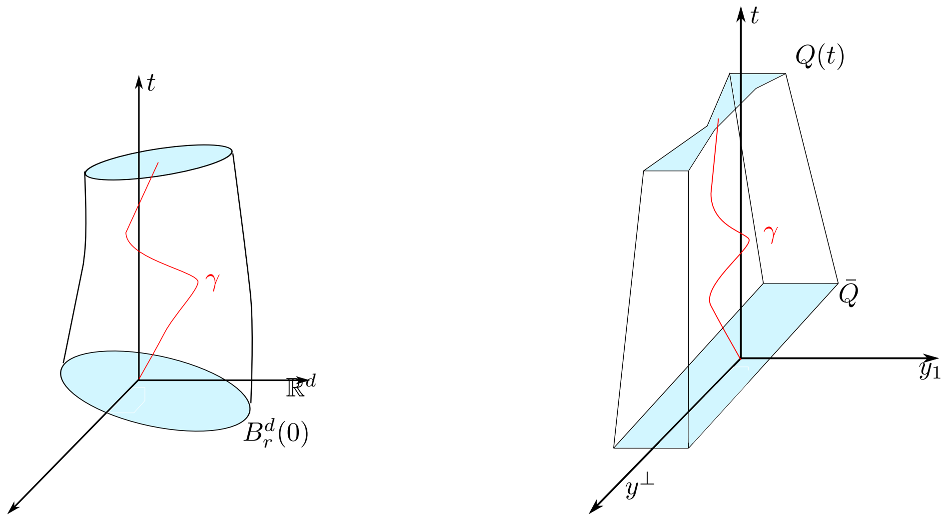
1.3.1. *The* BV *nearly incompressible case.* The last part of this work aims to give an interesting example where the above construction can be performed: it is the case where $\mathbf{b} \in L^1_t(\mathrm{BV}_x)_{\mathrm{loc}}$. In view of Main Theorem 2, without loss of generality, we assume $\rho = 1$ so that the vector field under consideration is exactly $(1, \mathbf{b})\mathcal{L}^{d+1}$ and we denote by $D\mathbf{b} = \int D\mathbf{b}(t)\, dt$ the derivative of $\mathbf{b}$. The construction considers a local approximating vector field for which the flow is Lipschitz and whose cylinders of flows satisfy Assumption 1.3.

For any matrix A and γ characteristic (and for $\ell, \delta_1 > 0$), one can define the cylinder
$$\phi_\gamma^{\ell,\delta_1}\big(t, \gamma(t) + e^{At}y\big) = \left[1 - \frac{1}{\delta_1 \ell} \operatorname{dist}\big(y, B^d_\ell(0)\big)\right]^+.$$

By some computations (similar to the renormalization analysis) one can to show that the integral (1.11) can be estimated by
$$\int \frac{1}{\sigma^{\ell,\delta_1}} \left[\int_{t_\gamma^-}^{t_\gamma^+} \int \big|(1, \mathbf{b}) \cdot \nabla_{t,x} \phi_\gamma^{\ell,\delta_1}\big| \, \mathcal{L}^{d+1} \right] \eta(d\gamma) \le C |D\mathbf{b} - A\mathcal{L}^{d+1}|\big(B^{d+1}_r(\bar t, \bar x)\big). \tag{1.13}$$

where σ^{ℓ,δ_1} is a suitable normalization constant and C a positive constant. In particular, the right hand side of (1.13) can be made arbitrarly small in the absolutely continuous part of $D\mathbf{b}$. Roughly speaking we have replaced the real evolution (under the flow of $\mathbf{b}$) of a ball $B^d_\ell(0)$ with an ellipsoid, given by a fixed matrix A (compare with Figure 3a): the computation behind (1.13) shows that the difference between the two evolutions (i.e. the lateral flux through the cylinder) is small, when compared to the volume of $B^d_\ell(0)$.

The estimates for the singular part are more delicate and depend heavily on the shape of the approximate cylinders of flow. The main idea is to choose properly the (non-transversal) sides' lenghts of the cylinders, in such a way to cancel the effect of the divergence. First of all, by Rank One Theorem, we can find a suitable (local) coordinate system $\mathbf{y} = (y_1, y^\perp) \in \mathbb{R}^d$ in which the derivative $D\mathbf{b}$ is essentially directed



(a) In the absolutely continuous part of $D\mathbf{b}$ the cylinders evolve under a constant matrix A, which will be taken close to $D^a\mathbf{b}$.

(b) The singular case: the cylinders shrink (if $\operatorname{div}\mathbf{b} < 0$) in a controlled way, their sides being graph of monotone Lipschitz functions which satisfy suitable differential equations.

Figure 3. Approximate cylinders of flow in the $L^1_t(\mathrm{BV}_x)_{\mathrm{loc}}$ nearly incompressible case.

toward a fixed direction (without loss of generality, the one given by $\mathbf{e}_1$). Accordingly, we define the (section at time t of the) cylinder

$$Q = Q_{\ell^{\pm}_{1,\gamma},\ell}(t) := \gamma(t) + \Big\{ \mathbf{y} = (y_1, y^\perp) : -\ell^-_1(t, y^\perp) \le y_1 \le \ell^+_1(t, y^\perp),\ |y^\perp| \le \ell \Big\}, \tag{1.14}$$

where $\ell > 0$ and $\ell^{\pm}_{1,\gamma}$ are monotone Lipschitz functions to be chosen. This is indeed a crucial step: we show it is possible to adapt locally the cylinders of approximate flows, by imposing that the sides' lengths $\ell^{\pm}_{1,\gamma}(t)$ are monotone functions satisfying suitable differential equations (see Figure 3b). In a simplified setting, i.e. if the level set of $b_1(t)$ were exactly of the form $y_1 = \text{constant}$, then we would impose

$$\frac{d}{dt}\ell^+_{1,\gamma}(t) = (Db_1)\big(\gamma(t), \gamma(t) + \ell^+_{1,\gamma}(t)\big) \tag{1.15}$$

(and an analogous relation for $\ell^-_{1,\gamma}$). Plugging the solution of (1.15) into the definition of the cylinder (1.14), we can show that the flux of $\mathbf{b}$ through the lateral boundary of Q is under control. Actually, a technical variation of this is needed in order to take into account the fact that the level sets are not of the form $y_1 = $ constant: to do this we exploit Coarea Formula and a classical decomposition of finite perimeter sets into rectifiable parts (De Giorgi's rectifiability Theorem). We show that, up to a $|D^{\mathrm{sing}}\mathbf{b}|$-small set, one can find Lipschitz functions $y_1 = L_{t,h}(y^\perp)$ in a fixed set of coordinates $(y_1, y^\perp) \in \mathbb{R} \times \mathbb{R}^{d+1}$, whose graphs cover a large fraction of the singular part $D^{\mathrm{sing}}\mathbf{b}\llcorner_{B^{d+1}_r(\bar t,\bar x)}$. We can at this point reverse the procedure, i.e. we construct a vector field starting from the level sets: this yields a BV vector field $\mathcal{U}(t)$ whose component $\mathcal{U}_1$ can be put into the right hand side of (1.15) and we can now perform the precise estimate of the flux of $\mathbf{b}$ through the lateral boundary of Q. By an application of the Radon-Nikodym Theorem, it follows that on a large compact set it holds that the flow integral (1.11) is controlled by $\tau|D^{\mathrm{sing}}\mathbf{b}|(B^{d+1}_r(\bar t,\bar x))$. Finally a covering argument implies that the measure ζ^C_τ can be taken to be $\tau|D\mathbf{b}|$, i.e. Theorem 1 holds.

1.4. **Structure of the paper.** The paper is organized as follows.

Section 2 introduces the main notations used in the paper.

Section 3 collects some results which will be used. After specifying the problem under consideration and observing that due to the locality of the result it is enough to consider vector fields with compact support, in Section 3.1 we recall the basic results on the existence of a Lagrangian representation η (Definition 3.1), i.e. a measure on the space of trajectories of the vector field $\mathbf{b}$ defined in an open interval. The only variation w.r.t. the results of [Smi94] is that, thanks to the form of the vector field (i.e. $(1, \mathbf{b})$), can parametrize the curves with t. The fact that η-a.e. curve is of finite length implies that

$$\int \bigg[\int_{t_\gamma^-}^{t_\gamma^+} |\dot\gamma(t)|\, dt \bigg] \eta(d\gamma) = \|\mathbf{b}\|_{L^1},$$

i.e. there exists the limit points $\gamma(t_\gamma^\pm)$: in particular for us $\mathrm{Graph}\,\gamma$ is the graph of γ together with its starting and ending points. Section 3.2 deals with a duality result which will lead to the untangling properties of the representation η: the fundamental result (Proposition 3.3) is that if B is a Borel set, and μ_i, $i = 1, 2$, are two bounded measures (possibly with different mass), then the dual of the optimal transport problem

$$\sup_{(\mathtt{p}_i)_\sharp \pi \leq \mu_i} \pi(B) = \inf \bigg\{ \sum_i \int h_i\, \mu_i, h_i \text{ Borel}, \sum_i h_i \geq \mathbb{1}_B \bigg\}$$

has a minimum, which is actually given by some characteristic functions $h_i = \mathbb{1}_{B_i}$.
Finally Section 3.3 recalls some fundamental properties of BV functions, which are used in the last part of the paper when studying the $L^1_t(\mathrm{BV}_x)$-case: the most important ones are the Coarea Formula, Theorem 3.7, and the Rank-One Property, Theorem 3.8.

The rest of the paper is divided into 3 parts, each studying a different problem: select suitable sets which can be used for testing purposes (Part 1), deduce from a local estimate on the Lagrangian representation some global uniqueness properties (Part 2), and finally show that $\mathbf{b} \in L^1_t(\mathrm{BV}_x)_{\mathrm{loc}}$ satisfies this local estimate (Part 3).

In Section 4 we give a property of open sets Ω with sufficiently regular boundary which (at the end) will imply that the normal trace controls the flow of trajectory across $\partial\Omega$: the idea is that there are two Lipschitz functions $\phi^{\delta,\pm}$ such that $\rho|(1, \mathbf{b}) \cdot \nabla \phi^{\delta,\pm}|\mathcal{L}^{d+1}$ converges to the outer/inner normal trace. Definition 4.1 of $\rho(1, \mathbf{b})$-*proper sets* requires more conditions, which are not restrictive being the sets used for testing purposes; subsequent remarks are addressed to possible extensions. The main results of Section 4.1 are that there are sufficiently many sets which have a simple geometry and are $\rho(1, \mathbf{b})$-proper (Lemma 4.8) and a condition to construct new proper sets (Proposition 4.9).
In particular, these sets can be perturbed: Section 4.2 construct indeed perturbations Ω_ε of proper sets Ω, which are still proper, arbitrarily close to the original Ω and such that the entering/exiting fluxes mainly occur across finitely many time-constant planes, Theorem 4.16.

The restriction operation $\rho(1, \mathbf{b})\mathcal{L}^{d+1} \mapsto \rho(1, \mathbf{b})\mathcal{L}^{d+1} \llcorner_\Omega$ has as a key point the computation of traces: indeed the boundary of $\partial\Omega$ adds a trace, i.e. a source/sink for the trajectories γ. The idea is to consider the operator

$$\gamma^{-1}(\Omega) = \bigcup_i (t_\gamma^{i,-}, t_\gamma^{i,+}), \qquad \mathtt{R}^i_\Omega \gamma = \gamma \llcorner_{(t_\gamma^{i,-}, t_\gamma^{i,+})},$$

and the natural image measure

$$(\mathtt{R}_\Omega)_\sharp \eta = \sum_i (\mathtt{R}^i_\Omega)_\sharp \eta.$$

After recalling some known results of (now) classical trace theory for L^∞-divergence measure vector fields, we show in Section 5.1 that the maps $t_\gamma^{i,\pm}$ are Borel (Lemma 5.7), and give a representation of the distributional trace as a countable sum of measures $(\mathtt{T}^{i,\pm}_\Omega)_\sharp \eta$ (Lemma 5.8); an example showing that the trace in general is not absolutely convergent is given in Example 5.9. A consequence of this lack of convergence is that $(\mathtt{R}_\Omega)_\sharp \eta$ is not a Lagrangian representation of $\rho(1, \mathbf{b})\mathcal{L}^{d+1}\llcorner_\Omega$, since the balance of the divergence is not true. By increasing the regularity of the vector field, one can obtain an absolutely convergent sequence of measures representing the trace for Lipschitz sets: this is the case of BV or BD vector fields, and this is studied in Section 5.2. After a trivial extension to $L^1(\mathrm{BD}_x)_{\mathrm{loc}}$ of the chain rule for traces (Proposition 5.11), in Proposition 5.12 we show that $(\mathtt{R}_\Omega)_\sharp \eta$ is an absolutely convergent sum of measures.

The main aim of this analysis is to identify two disjoint subsets $A^\pm$ of $\partial\Omega$ such that $(\mathtt{T}^{i,\pm}_\Omega)_\sharp \eta$ is concentrated on $A^\pm$: this is exactly the case of proper sets, and it is studied in Section 6. The key fact, used several times in the section, is that the trace controls the flux of trajectories across $\partial\Omega$: using the perturbations Ω^ε of Section 4.2 one can further show that a weak differentiability holds, Corollary 6.6, and finally that $(\mathtt{R}_\Omega)_\sharp\eta$ is a Lagrangian representation of $\rho(1,\mathbf{b})\mathcal{L}^{d+1}\llcorner\Omega$, Theorem 6.8. Further regularity properties and stability w.r.t. perturbations are analyzed in Corollary 6.9 and Proposition 6.10.

This concludes Part 1, and next we begin with Part 2.

The starting point of Section 7 is to give a set of assumptions in a proper set Ω which implies uniqueness up to a set of trajectories whose η-measure is controlled: this is mainly Assumption 7.1 of Section 7.1. (See also Remark 7.5 for essentially equivalent conditions, Assumption 7.6). This uniqueness result is divided into two propositions: in the first, Proposition 7.2, we control the trajectory which starts from the same point of the boundary and subsequently bifurcate. In Proposition 7.3 instead we use transference plans to control the amount of trajectories starting from two different points of $\partial\Omega$ and intersecting at a later time. Example 7.4 shows that Proposition 7.2 is sharp, in the sense that in general one cannot hope for a full control of transference plans π between η^{in}.
Restricting to the crossing trajectories, instead, it is possible to have such a control: indeed, Corollary 7.8 gives the estimate on the π-measure of trajectories starting from two different points and intersecting, while at the expenses of $o(1)\mu^-(\Omega)$, Proposition 7.11 considers the other case. The final result is Theorem 7.14, which follows from the above two bounds and Kellerer's duality results: it states that, after removing a set of trajectories whose η-measure is explicitly controlled, the remaining curves are either disjoint or one a subset of the other. For convenience, the analysis is performed first on perturbation of propers sets, and then passed to the limit as shown in Theorem 7.15.

The next section, Section 8, addresses the problem of passing the local results obtained in Section 7 to a global estimate. From the estimates of Theorem 7.15, it is natural to introduce the untangling functionals $\mathfrak{f}^{\mathrm{in}}$ and $\mathfrak{f}^{\mathrm{out}}$, Definitions 8.1 and 8.2. The main result of Section 8.1 is their subadditivity, Proposition 8.3. Lemma 8.4 gives a simple but useful estimate on their relationship.
Being the untangling functionals subadditive, it is natural to control their values with a measure: this is exactly Assumption 8.5, Section 8.2. A standard covering argument yields Theorem 8.9, which is the global version of Theorem 7.15. In the case the comparison measure can be made arbitrarily small (which will be the case if Assumption 8.12 is satisfied) then Corollary 8.11 shows that the Lagrangian representation η is *untangled*, Definition 8.10.

The last section of this second part shows that if η is untangled then, as in the classical case, there exists a partition of $\mathbb{R}^{d+1}$ into trajectories of $\mathbf{b}$ (Proposition 9.1) such that the PDE $\mathrm{div}(\rho(1,\mathbf{b}))=\mu$ can be decomposed into ODEs on the characteristics (Proposition 9.3). The final result of Section 9.1 is the existence of a partition via characteristics, Definition 9.4 and Theorem 9.5.
In the second part, Section 9.2, we prove how a partition via characteristics is the same in the class $\rho' \in L^\infty(\rho)$, Theorem 9.6; this allows the explicit computation of the chain rule, performed in Proposition 9.8.

The last part, Part 3, shows that $\mathbf{b} \in L^1_t(\mathrm{BV}_x)_{\mathrm{loc}}$ satisfies Assumption 8.12.

In Section 10, we exploit Coarea formula and Rank-One Theorem to show that it is possible to approximate locally the singular part of $D\mathbf{b}$ with a measure concentrated on uniformly Lipschitz graphs, Corollary 10.5. The proof of this fact is split into several steps (Propositions 10.2-10.3 and Corollary 10.4) and relies ultimately on the properties of sets of finite perimeter, in particular the De Giorgi Rectifiability Theorem. Using the Rank one property, the vector valued case is reduced to the previous analysis and we obtain the desired decomposition in Corollary 10.5.

The key section is Section 11, where the explicit form of the local cylinders is exhibited. We have to consider two cases.
In the absolutely continuous part of $D\mathbf{b}$, Section 11.1, one compares the Lagrangian flow η with the linear flow generated by a constant matrix A. In this case, the analysis is pretty much similar to the standard renormalization estimates, giving in Proposition 11.1 the correct bounds.
The singular part (Section 11.2) is definitely more involved: the cylinders are constructed by solving a PDE (equations (11.5)) using the approximate vector field constructed in the previous section. Lemma 11.2 guarantees that the Lipschitz regularity of sets is preserved in time, so that they can be used as approximate cylinders of flows. Lemmata 11.3, 11.4 estimates the lateral flows of these cylinders and yield Proposition 11.5, giving the correct bound for the singular case. We thus conclude with Theorem 11.6, which states that Assumption 8.12 holds for $\mathbf{b} \in L^1_t(\mathrm{BV}_x)_{\mathrm{loc}}$.

We collect in the last section, Section 12 the proof of Lemmata 11.3, 11.4.

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2. Notation

The aim of this section is to introduce some basic notation, used through the paper.

The real numbers will be denoted by $\mathbb{R}$, the d-dimensional real vector space by $\mathbb{R}^d$, and its norm by $|\cdot|$. In the following we will consider the space $\mathbb{R}^{d+1}$, whose coordinates will be denoted by t time and x space, with $t \in \mathbb{R}, x \in \mathbb{R}^d$.

The open ball in $\mathbb{R}^d$ centered in x with radius r is

$$B^d_r(x) = \big\{y : |y - x| < r\big\}.$$

The unit sphere of dimension d will be denoted by $\mathbb{S}^d$. Its $\mathcal{H}^d$-measure is $d\omega_d$, so that

$$\mathcal{L}^d(B^d_r(0)) = \omega_d r^d.$$

More generally, in a metric space X the ball centered in $\mathtt{x} \in X$ with radius r will denoted $B^X_r(\mathtt{x})$, and $B_r(\mathtt{x})$ when no confusion occurs about X. A generic open set in $\mathbb{R}^d$ will be Ω. The norm in a generic Banach space will be denoted by $\|\cdot\|$, with an index referring to the space whenever some confusion may occur.

The closure of a set A is denoted by $\operatorname{clos} A$, usually being clear the ambient topological space. The relative closure of A in the topological space B is $\operatorname{clos}(A, B)$. The interior is similarly written as $\operatorname{int} A$ or $\operatorname{int}(A, B)$, and the frontier $\operatorname{Fr}(A, B)$. In same cases (mainly for $\Omega \subset \mathbb{R}^d$) we will use the more standard notation $\partial\Omega$. We will say that $A \Subset B$ if $\operatorname{clos} A$ is a compact set contained in B. A neighborhood of $\mathtt{x} \in X$ will be written as $U_{\mathtt{x}}$.

We denote the projection on the space X by $\mathtt{p}_X$: in general the product space $X \times Y$ is clear from the context, and sometimes we will also write $\mathtt{p}_j : \prod_i X_i \to X_j$ as the projection on the j-component X_j. In the product space $X \times Y$, for all sets A we will use the notation

$$A(x) = \big\{y : (x, y) \in A\big\}, \quad A(y) = \big\{x : (x, y) \in A\big\}.$$

We will be not very coherent when writing also $A(\bar{x}) = A \cap \{x = \bar{x}\} \subset X \times Y$, and $A(x) = A_x \subset Y$. If A is a set, we will denote by $\mathbb{1}_A$ the characteristic function

$$\mathbb{1}_A(x) := \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

We say that the family A_α is a covering of A if

$$A \subset \bigcup_\alpha A_\alpha.$$

They are a partition if they are a disjoint covering, i.e. $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$.

By the bold letters we denote some particular vectors, e.g. $\mathbf{b} = (b_i)^d_{i=1}$ or n. For notational convenience, we will write

$$x_{\mathbf{n}} = (x \cdot \mathbf{n})\mathbf{n}, \qquad x^\perp_{\mathbf{n}} = x - x_{\mathbf{n}}.$$

Often we will identify each of these vectors with their subspace vectors, e.g. $x_{\mathbf{n}} \simeq x \cdot \mathbf{n}$. A generic vector will be written as B: we will sometime use this notation when the particular structure of B is not important. If the vector field $\mathbf{b} : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is time dependent, then we will use either the notation $\mathbf{b}(t)$ or $\mathbf{b}_t$.

A distribution f evaluated on a function ψ will be written as

$$\int f\psi\, \mathcal{L}^d \qquad \text{or} \quad \langle f, \psi\rangle.$$

The distributional partial derivatives of a function f will be written as $\partial_t f_t$, $\partial_{x_i} f$. The differential of a smooth function f will be written as Df, and the divergence of a vector field $\mathbf{b}$ by $\operatorname{div} \mathbf{b}$. For time dependent functions and vectors we will consider only space gradients and divergence,

$$Df(t) = \big(\partial_{x_i} f_i(t, x_1, \dots, x_d)\big)_{i=1,\dots,d}, \qquad \operatorname{div} \mathbf{b}(t) = \sum_{i=1}^d \partial_{x_i} \mathbf{b}_i(t, x_1, \dots, x_d).$$

If $\mathbf{e}$ is a unit vector, the derivative of $\mathbf{b}$ along $\mathbf{e} = (e_i)_{i=1}^d$ will be

$$D_{\mathbf{e}} f = \sum_{i=1}^d e_i \partial_i f.$$

In general, we will use the same notation also for the distributional counterparts of Df, $\operatorname{div} \mathbf{b}_t$, $D_{\mathbf{e}} f$. The space of functions f which belong $\mathrm{BV}(B^d_r(0))$ for every $r > 0$ is denoted by $\mathrm{BV}_{\mathrm{loc}}$.

When the function is defined in a subset of the ambient space, the domain will be written as $\mathcal{D}(f)$, while the range is $\mathcal{R}(f)$. The graph of a function f is denoted as $\operatorname{Graph} f$, and the support of by $\operatorname{supp} f$.

We will use the notation $\mathcal{L}^d$ for Lebesgue measure in $\mathbb{R}^d$, $\mathcal{H}^d$ for the d-dimensional Hausdorff measure and δ_x as the Dirac mass at x. For a generic signed Radon measure μ on $\mathbb{R}^d$ we will write $|\mu|$ as its total variation. Given another Radon measure ν, the Radon-Nikodym derivative of ν w.r.t. the positive measure $\mu \geq 0$ will be written as $\frac{d\nu}{d\mu}$. We say that ν, μ are orthogonal and we write $\mu \perp \nu$ if there exists two disjoint sets A_1, A_2 such that $A_1 \cup A_2 = \mathbb{R}^d$ and $|\mu|(A_2) = |\nu|(A_1) = 0$. The set of signed Radon measures over X is denoted by $\mathcal{M}(X)$, the positive Radon measures with $\mathcal{M}^+(X)$ and the bounded Radon measures by $\mathcal{M}_b(X)$.

Usually the integral of a Borel function f w.r.t. a measure μ will be written (when it exists) as

$$\int f \, \mu$$

and in the case $\mu = \mathcal{L}^d$ as

$$\int f \, \mathcal{L}^d \qquad \text{or} \qquad \int f \, dx.$$

We will also use the standard notations $L^1(\mu, Y)$ ($L^\infty(\mu, Y)$) for the space of functions with values in Y Banach whose norm is μ-integrable (μ-essentially bounded), and $C(X, Y)$ for the space of continuous functions. If $X = \mathbb{R}^d$, $C^k(\mathbb{R}^d, \mathbb{R}^{d'})$ is the space of functions with continuous partial derivatives up to order k. Whenever the measure $\mu = \mathcal{L}^d$ and $Y = \mathbb{R}$ we will just write $L^1(\mathbb{R}^d)$ ($L^\infty(\mathbb{R}^d)$), and we add in case the index loc to denote properties which holds locally, e.g. local integrability (local boundedness). Sometimes, when the space Y is clear from the context, we use the notation $L^1(\mu)$ ($L^\infty(\mu)$).

The disintegration of a measure μ w.r.t. a partition $\{A_\alpha\}_\alpha$ will be written as

$$\mu = \int \mu_\alpha \, f_\sharp \mu(d\alpha),$$

where f is the partition function, i.e. $f^{-1}(\alpha) = A_\alpha$. Usually f is a projection.

The restriction of a measure μ to a set A will be written as $\mu \llcorner_A$, and similarly for a function $f \llcorner_A$. Recall also the decomposition

$$\nu = \frac{d\nu}{d\mu} \mu + \nu^\perp,$$

where $\nu^\perp$ is orthogonal to μ. If $\nu^\perp = 0$ then ν is a.c. w.r.t. μ and we will write $\nu \ll \mu$. When the measure $\mu = \mathcal{L}^d$, then the first term will be denoted by $\nu^{\mathrm{a.c.}}$ (either for the function or the measure). Since all results in this paper are local in space-time, we will not distinguish between weak and narrow convergence, and sometime we will just write weak (or weak*) convergence of measures to denote both of them.

For a Borel function f, set $f_\sharp \mu$ as the push-forward of μ through f, i.e. for g Borel

$$\int g \, (f_\sharp \mu) = \int (g \circ f) \, \mu,$$

where $g \circ f$ denotes the usual composition of the two functions f, g. Note that we will sometime avoid to write the set of integration, being implicitly characterized by the measure w.r.t. we are integrating.

Given a point $x \in \mathbb{R}^d$ and $r > 0$, let f^r_x be the rescaling of f about $x \in \mathbb{R}^d$,

$$f^r_x(y) := f(x + ry).$$

For a measure μ, similarly we define μ^r_x as

$$\int f(y) \, \mu^r_x(dy) = \int f\left(x + \frac{y - x}{r}\right) \mu(dy).$$

In the case of 1-dimensional BV functions f (or, in general, whenever the limits exist), we will write

$$f(\bar{x}\pm) = \lim_{x \to \bar{x}} f(x)$$

for the right/left limit.

The average integral on a set will be written as

$$\fint_A f\,\mu := \frac{1}{\mu(A)} \int_A f\,\mu.$$

A smooth positive function φ with compact support and $\mathcal{L}^d$-integral equal to 1 will be called convolution kernel. We will use the notation $\varphi^\varepsilon = \varepsilon^{-d}\varphi_0^{1/\varepsilon}$. A smooth test function will be denoted by ψ. We will denote the convolution in $\mathbb{R}^d$ by $*$.

The notation ϕ is usually reserved to some particular functions, and will have some apex/index depending on the case. The same for particular sets Q. Moreover, when $Q \subset \mathbb{R} \times \mathbb{R}^d$, we will denote $\partial^l Q$ the boundary of Q in the open set $\{t \in \mathrm{int}(\mathtt{p}_1 Q)\}$.

An open set $\Omega \subset \mathbb{R}^d$ is said to be *Lipschitz* or *Lipschitz regular* if $\partial\Omega$ is *Lipschitz*: the latter means that for every point $x \in \partial\Omega$ there exists a Lipschitz function $\varsigma_x : \mathbb{R}^{d-1} \supset U_x \to \mathbb{R}$ and $r > 0$ such that in a local coordinate system

$$\partial\Omega \cap B^d_r(x) = \mathrm{Graph}(\mathrm{id}, \varsigma_x).$$

The notation id is for the identity function $\mathrm{id}(x) = x$.

To conclude this section, we will use L for a scale constant, C_d for a dimensional constant and C for a generic constant. If f is some function, we will write $\mathcal{O}(f)$ for a quantity equivalent to f or $o(f)$ for an infinitesimal quantity w.r.t. f: usually the point about where the limit are taken is clear from the context. A negligible set (w.r.t. some given measure) is often denoted by N.

3. Preliminaries

In this section we recall some basic results concerning measure divergence vector fields, their representation as superimposition of curves [Smi94], duality for transportation problems [Kel84] and properties of BV functions [AFP00].

Consider a vector field with compact support of the form

$$\rho(1, \mathbf{b}) \in L^1(\mathbb{R}^{d+1}, \mathbb{R}^{d+1}),$$

where

$$\rho : \mathbb{R}^{d+1} \to \mathbb{R}^+, \qquad \mathbf{b} : \mathbb{R}^{d+1} \to \mathbb{R}^d.$$

We assume moreover that it holds in the sense of distribution,

$$\mathrm{div}(\rho(1, \mathbf{b})) = \mu \in \mathscr{M}(\mathbb{R}^{d+1}), \tag{3.1}$$

which means that $\rho(1, \mathbf{b})$ is a *measure-divergence vector field*. To avoid dealing with sets of $\mathcal{L}^d$-negligible measure, we will assume that $\rho, \mathbf{b}$ are defined pointwise as Borel functions.

An absolutely continuous curve $\gamma : I_\gamma \to \mathbb{R}^d$, where $I_\gamma = (t^-_\gamma, t^+_\gamma)$ is a time interval, is a *characteristic of the vector field* $\rho(1, \mathbf{b})$ if it solves the ODE

$$\frac{d}{dt}\gamma(t) = \mathbf{b}(t, \gamma(t)),$$

where the equivalence holds $\mathcal{L}^1$-a.e. in I_γ.

3.1. Lagrangian representations.

We consider the space of curves γ: more precisely, let

$$\varUpsilon = \{t_1 < t_2\} \times C(\mathbb{R}, \mathbb{R}^d),$$

and its subset made of characteristics

$$\varGamma = \Big\{(t_1, t_2, \gamma) \in \varUpsilon : \gamma \text{ characteristic in } (t_1, t_2)\Big\}.$$

One can show that $\varGamma$ is a Borel subset of $\varUpsilon$. We will often consider γ as defined in the interval $I_\gamma = (t^-_\gamma, t^+_\gamma)$, i.e. $\gamma = \gamma\llcorner_{(t^-,t^+)}$.

Definition 3.1. We say that a bounded, positive measure $\eta \in \mathcal{M}^+_b(\varUpsilon)$ is a *Lagrangian representation of the vector field* $\rho(1, \mathbf{b})\,\mathcal{L}^{d+1}$ if the following conditions hold:

(1) η is concentrated on the set Γ of absolutely continuous solutions to the ODE

$$\dot\gamma(t) = \mathbf{b}(t, \gamma(t)), \tag{3.2}$$

which explicitly means for every $s, t \in I_\gamma$

$$\int_\Gamma \left| \gamma(t) - \gamma(s) - \int_s^t \mathbf{b}(\tau, \gamma(\tau))\, d\tau \right| \eta(d\gamma) = 0;$$

(2) if $(\mathrm{id}, \gamma)\colon I_\gamma \to I_\gamma \times \mathbb{R}^d$ denotes the map defined by $t \mapsto (t, \gamma(t))$, then

$$\rho(1, \mathbf{b})\, \mathcal{L}^{d+1} = \int_\Gamma (\mathrm{id}, \gamma)_\sharp \big((1, \dot\gamma) \mathcal{L}^1 \big)\, \eta(d\gamma);$$

(3) we can decompose the divergence μ as local superposition of Dirac masses without cancellation, i.e.

$$\begin{aligned} \mu &= \int_\Gamma \Big[\delta_{t^-_\gamma, \gamma(t^-_\gamma)}(dt, dx) - \delta_{t^+_\gamma, \gamma(t^+_\gamma)}(dt, dx) \Big]\, \eta(d\gamma), \\ |\mu| &= \int \Big[\delta_{t^-_\gamma, \gamma(t^-_\gamma)}(dt, dx) + \delta_{t^+_\gamma, \gamma(t^+_\gamma)}(dt, dx) \Big]\, \eta(d\gamma), \end{aligned}$$

where we recall that, for every γ, the interval in which it is a characteristic is denoted by $(t^-_\gamma, t^+_\gamma) = I_\gamma$.

The existence of such a measure η follows from the analysis of [Smi94], where the more general case of 1-dimensional normal currents is addressed (see [Smi94] for the classical results for currents in Euclidean space and [PS12, PS13] for an extension to Ambrosio-Kirchheim currents): the only difference is the time parametrization. Indeed, consider any curve $\tau \mapsto (t(\tau), \gamma(\tau))$ solving the normalized ODE (3.2),

$$\frac{dt}{d\tau} = \frac{1}{|(1, \mathbf{b})|}, \qquad \frac{d\gamma}{d\tau} = \frac{1}{|(1, \mathbf{b})|} \mathbf{b}(t(\tau), \gamma(\tau)) = \frac{dt}{d\tau} \mathbf{b}(t(\tau), \gamma(\tau)).$$

Define now $s = s(t)$ as the inverse of the Lipschitz function $t \mapsto t(\tau)$ (s is strictly increasing) then $t \mapsto \gamma(s(t))$ is continuous. From Coarea formula we deduce that the following change of variables holds:

$$\int_{t_1}^{t_2} |\mathbf{b}(t, \gamma(s(t)))|\, dt = \int_{s(t_1)}^{s(t_2)} |\mathbf{b}(t(\tau), \gamma(\tau))| \frac{dt}{d\tau}\, d\tau$$

so that in particular $\mathbf{b}(t, \gamma(s(t))) \in L^1_{\mathrm{loc}}$ (being η-a.e. of finite length) and again by Coarea without modulus $\dot\gamma(s(t)) = \mathbf{b}(t, \gamma(s(t)))$.

Observe that for all γ the interval of definition is a bounded time interval (recall that we assume $\rho(1, \mathbf{b})\, \mathcal{L}^{d+1}$ with compact support), so that if $\mu^\pm$ is the positive/negative part of the divergence we can disintegrate η according to

$$\eta = \int \eta_z\, \mu^-(dz), \qquad \mu^\pm = (t^\pm_\gamma)_\sharp \eta. \tag{3.3}$$

Notice that γ can be defined in the closed or open interval: adding or subtracting the end points does not change the representation.Nevertheless, when we need to study some sets, we will consider the graph of γ in the closed interval, i.e. with a slight abuse of notation

$$\mathrm{Graph}\, \gamma = \mathrm{clos}\, \mathrm{Graph}\, \gamma. \tag{3.4}$$

We remark finally that, by the first and second points of Definition 3.1, it follows that

$$\int_\Gamma \bigg[\int_{I_\gamma} |\dot\gamma|\, \mathcal{L}^1 \bigg] \eta(d\gamma) = \int_\Gamma \bigg[\int_{I_\gamma} |\mathbf{b}(t, \gamma(t))|\, dt \bigg] \eta(d\gamma) = \int \rho |\mathbf{b}| \mathcal{L}^{d+1},$$

so that the total variation of η-a.e. curve is finite, and thus $\gamma(t^\pm_\gamma) \in \mathbb{R}^d$ exists.

3.2. **Optimal transport and duality.** In this section we recall some results contained in the paper [Kel84]. Given finitely many finite measures $\mu_i \geq 0$ over Polish spaces X_i, define the set of *admissible transference plans* $\mathsf{Adm}(\mu_i)$ defined by

$$\mathsf{Adm}(\mu_i) = \Big\{\pi \geq 0 : (\mathtt{p}_i)_\sharp \pi \leq \mu_i\Big\} \subset \mathcal{M}^+\bigg(\prod_i X_i\bigg).$$

Given a positive Borel function $h \geq 0$, consider the following duality problem:

$$\sup_{\mathsf{Adm}(\mu_i)} \int h\,\pi = \inf\bigg\{\sum_i \int h_i\,\mu_i, h_i \text{ Borel}, \sum_i h_i \geq h\bigg\}. \tag{3.5}$$

The following observations are fairly easy.

(1) One can get rid of the fact that the measures μ_i have different mass by adding an additional $\bar{x}_i$ point to X_i and putting a Dirac delta in x_i of suitable mass. Extending h to $\prod_i X_i \sqcup \{\bar{x}_i\}$ by putting $h = 0$ on the set $\cup_i \mathtt{p}_i^{-1}(\bar{x}_i)$, the values of the two sides of (3.5) is unchanged.
(2) Clearly the first side is maximized when $(\mathtt{p}_i)_\sharp \pi = \mu_i$, because of the positivity of the measures.

We thus are in the setting considered by Kellerer, and we can thus state the following.

Theorem 3.2 (Theorems 2.14,2.12 of [Kel84])**.** *The equality (3.5) holds if h is a Borel function, and the infimum is actually a minimum.*

Moreover, in the case of 2 factors X_1, X_2 and when h is a characteristic function, the infimum can be restricted to Borel sets.

Proposition 3.3 (Proposition 3.3. of [Kel84])**.** *If $n = 2$ and $h = \mathbb{1}_B$, then the r.h.s. of (3.5) can be replaced by*

$$\inf\bigg\{\sum_{i=1,2} \mu_i(B_i), B_i \textit{ Borel}, \sum_{i=1,2} \mathbb{1}_{B_i} \geq \mathbb{1}_B\bigg\},$$

and the minimum is attained.

3.3. BV **and** BD **functions.** For $\mathbf{b} \in L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ we denote by $D\mathbf{b} = (D_i b^j)_{i,j}$ the derivative in the sense of distributions of $\mathbf{b}$, i.e. the $\mathbb{R}^{d\times d}$-valued distribution defined by

$$D_i b^j(\varphi) := \int_{\mathbb{R}^d} b^j \frac{\partial \phi}{\partial x_i}\,dx \qquad \forall \phi \in C^\infty_c(\Omega), \qquad 1 \leq i,j \leq d.$$

Furthermore, we denote by $E\mathbf{b} = (E_{ij}\mathbf{b})_{ij}$ the symmetric part of the distributional derivative of $\mathbf{b}$, i.e.,

$$E_{ij}\mathbf{b} := \frac{1}{2}(D_i b^j + D_j b^i), \qquad 1 \leq i,j \leq d.$$

Definition 3.4 (BV and BD functions)**.** We say that $\mathbf{b} \in L^1(\mathbb{R}^d; \mathbb{R}^d)$ has *bounded variation* in $\mathbb{R}^d$, and we write $b \in \mathrm{BV}(\mathbb{R}^d; \mathbb{R}^d)$ if $D\mathbf{b}$ is representable by a $\mathbb{R}^{d\times d}$-valued measure, still denoted with $D\mathbf{b}$, with finite total variation in $\mathbb{R}^d$. We say that $\mathbf{b} \in L^1(\mathbb{R}^d; \mathbb{R}^d)$ has *bounded deformation* in $\mathbb{R}^d$, and we write $b \in \mathrm{BD}(\mathbb{R}^d)$, if $E_{ij}\mathbf{b}$ is a Radon measure with finite total variation in $\mathbb{R}^d$ for any $i,j = 1,\dots,d$.

We now recall the following theorem.

Theorem 3.5 (Structure theorem for BV)**.** *If $\mathbf{b} : \mathbb{R}^d \to \mathbb{R}^d$ is a $\mathrm{BV}_{\mathrm{loc}}$ vector field, then the measure $D\mathbf{b}$ can be decomposed into three parts*

$$D\mathbf{b} = D^{\mathrm{a.c.}}\mathbf{b} + D^{\mathrm{sing}}\mathbf{b} = D^{\mathrm{a.c.}}\mathbf{b} + D^{\mathrm{cantor}}\mathbf{b} + D^{\mathrm{jump}}\mathbf{b},$$

where

(1) $D^{\mathrm{a.c.}}\mathbf{b}$ is the a.c. part of $D\mathbf{b}$ w.r.t. $\mathcal{L}^d$,
(2) $D^{\mathrm{sing}}\mathbf{b}$ is the singular part of $D\mathbf{b}$ w.r.t. $\mathcal{L}^d$,
(3) $D^{\mathrm{jump}}\mathbf{b}$ is the (d-1)-rectifiable part, absolutely continuous w.r.t. to the $\mathcal{H}^{d-1}$-measure concentrated on the (d-1)-countably rectifiable set J,
(4) $D^{\mathrm{cantor}}\mathbf{b}$ is the residual part, orthogonal to the Lebesgue measure, and such that each set with finite $\mathcal{H}^{d-1}$-measure is $D^{\mathrm{cantor}}\mathbf{b}$-negligible.

The jump set J is determined by the following property.

Proposition 3.6. *The blow-up* $\mathbf{b}^r_x$ *converges in* L^1 *to the pure jump*

$$\bar{\mathbf{b}} = \begin{cases} \mathbf{b}^- & x \cdot \mathsf{n} < 0, \\ \mathbf{b}^+ & x \cdot \mathsf{n} > 0. \end{cases} \tag{3.6}$$

The existence of the approximate limits $\mathbf{b}^\pm$ and of the normal vector n are part of the statement, and n is determined up to a sign. In particular it follows that

$$\lim_{r\to 0} \frac{(D\mathbf{b})^r_x}{r^{d-1}} = (\mathbf{b}^+ - \mathbf{b}^-)\,\mathsf{n}\,\mathcal{H}^{d-1} \llcorner_{\{x\cdot\mathsf{n}=0\}}.$$

In the case of scalar functions $f \in \mathrm{BV}(\mathbb{R}^d;\mathbb{R})$ the following Coarea formula holds. We denote the super level sets by $E_h := f^{-1}((h,+\infty))$. In the case the function f needs to be specified, we will write E^f_h.

Theorem 3.7 (Coarea)**.** *It holds*

$$|Df| = \int_{\mathbb{R}} |D\mathbb{1}_{E_h}|\,\mathcal{L}^1(dh), \qquad Df = \int_{\mathbb{R}} D\mathbb{1}_{E_h}\,\mathcal{L}^1(dh).$$

In the case of BV vector field $\mathbf{b}$, we recall the following deep result, due to Alberti:

Theorem 3.8 (Alberti's Rank-one)**.** *It holds*

$$D\mathbf{b} = M(x)|D^{\mathrm{a.c.}}\mathbf{b}| + \mathsf{n}(x) \otimes \mathsf{m}(x)|D^{\mathrm{sing}}\mathbf{b}|.$$

In the following we will use the notation n and m to denote the two unit vectors in the rank-one property. The matrix $M(x)$ will denote the Radon-Nicodym derivative of the absolutely continuous part. Note that from the orthogonality of the decomposition

$$|D^{\mathrm{a.c.}}\mathbf{b}| = |D\mathbf{b}|^{\mathrm{a.c.}}, \quad |D^{\mathrm{cantor}}\mathbf{b}| = |D\mathbf{b}|^{\mathrm{cantor}}, \quad |D^{\mathrm{jump}}\mathbf{b}| = |D\mathbf{b}|^{\mathrm{jump}}.$$

3.3.1. *Sets of finite perimeter.* Let $F \subset \mathbb{R}^d$: we will say it is of (locally) *finite perimeter* is the characteristic function $\mathbb{1}_F \in \mathrm{BV}(\mathbb{R}^d;\mathbb{R})$ (resp. locally of bounded variation). We recall that the *reduced boundary* $\partial^\star F$ of F is the set of points such that

$$\lim_{r\searrow 0} \frac{|D\mathbb{1}_F|(B^d_r(x))}{r^{d-1}} = \omega_{d-1}, \quad \lim_{r\searrow 0} \frac{D\mathbb{1}_F(B^d_r(x))}{|D\mathbb{1}_F|(B^d_r(x)} = \mathsf{n}(x),$$

where $\mathsf{n}(x)$ is the *measure theoretical inner unit normal.*

$$|D\mathbb{1}_F| = \mathcal{H}^{d-1} \llcorner_{\partial^\star F}.$$

Furthermore, it holds

$$\lim_{r\searrow 0} \frac{1}{\omega_d r^d} \mathcal{L}^d\big(F \cap B^d_r(x) \cap \{x \cdot \mathsf{n}(x) > 0\}\big) = 1, \quad \lim_{r\searrow 0} \frac{1}{\omega_d r^d} \mathcal{L}^d\big(F \cap B^d_r(x) \cap \{x \cdot \mathsf{n}(x) < 0\}\big) = 0.$$

We finally recall the following (see, for instance, [Zie89, Thm. 5.7.3]):

Theorem 3.9 (De Giorgi)**.** *If* $F \subset \mathbb{R}^d$ *is of locally finite perimeter, then* $\partial^\star F$ *is countably* $\mathcal{H}^{d-1}$ *rectifiable.*

Part 1

Proper sets, Lagrangian representations and traces

This part addresses to the following

Problem. Characterize the open sets $\Omega \subset \mathbb{R}^{d+1}$ such that, if η is a Lagrangian representation of $\rho(1,\mathbf{b})\,\mathcal{L}^{d+1}$ and $\mathtt{R}_\Omega\gamma = \gamma\llcorner_{(\mathrm{id},\gamma)^{-1}(\Omega)}$ is the restriction to Ω of a trajectory, then $(\mathtt{R}_\Omega)_\sharp\eta$ is a Lagrangian representation of $\rho(1,\mathbf{b})\,\mathcal{L}^{d+1}\llcorner_\Omega$.

More precisely, setting

$$(\mathrm{id},\gamma)^{-1}(\Omega) = \bigcup_{i\in\mathbb{N}}(t_\gamma^{i,-}, t_\gamma^{i,+}), \qquad \mathtt{R}^i_\Omega\gamma = \gamma\llcorner_{(t_\gamma^{i,-}, t_\gamma^{i,+})},$$

we can define the quantity

$$(\mathtt{R}_\Omega)_\sharp\eta = \sum_{i\in\mathbb{N}}(\mathtt{R}^i_\Omega)_\sharp\eta. \tag{3.7}$$

In general, the series in (3.7) does not converge; nevertheless, for a fixed vector field $\rho(1,\mathbf{b})$ it is possible to give sufficient conditions on Ω such that $(\mathtt{R}_\Omega)_\sharp\eta$ is well defined and provides a Lagrangian representation for $\rho(1,\mathbf{b})\,\mathcal{L}^{d+1}\llcorner_\Omega$. The sets which enjoy this property will be called $\rho(1,\mathbf{b})$-proper sets and they can be characterized by a strong trace approximation property.

The convergence of the series is indeed strongly related to the fact that it is possible to split the boundary $\partial\Omega$ into two disjoint sets $A^\pm$ where the trajectories are only entering or only exiting. Whenever the vector field has more regularity, one can relax the requests on proper sets: an important example is the class of BD vector fields, where the set can be taken merely Lipschitz.

4. Proper sets and their perturbations

This section is divided in two parts.

In the first one we define a family of sets which have good trace properties for a given vector field of the form $\rho(1,\mathbf{b}) \in L^1(\mathbb{R}^{d+1},\mathbb{R}^{d+1})$: we call these sets $\rho(1,\mathbf{b})$-*proper*. Their main properties are that their boundary $\partial\Omega$ is piecewise C^1, it is made of Lebesgue points of $\rho(1,\mathbf{b})$ and more importantly that the measure $\rho(1,\mathbf{b})\mathcal{H}^d\llcorner_{\partial\Omega}$ is the strict limit of the measures $\rho(1,\mathbf{b})\cdot\nabla_{t,x}\phi^{\delta,\pm}$, where

$$\phi^{\delta,+}(x) = \max\left\{1 - \frac{\mathrm{dist}(x,\Omega)}{\delta}, 0\right\}, \qquad \phi^{\delta,-}(x) = \min\left\{\frac{\mathrm{dist}(x,\mathbb{R}^{d+1}\setminus\Omega)}{\delta}, 1\right\}.$$

These conditions will be essential to show that $\rho(1,\mathbf{b})\mathcal{H}^d\llcorner_{\partial\Omega}$ is actually measuring the flow of $\rho(1,\mathbf{b})$ across $\partial\Omega$. Since these sets are used to test the vector fields, we will not consider the most general definition: we just want to have sufficiently many sets for testing purposes, and we prefer to avoid unnecessary technicalities.

In the second part of this section we perturb these sets in order to take advantage of the fact that the vector field has the form $(1,\mathbf{b})$: the idea is to have the inflow and outflow occurring on time-constant hyperplanes, i.e. regions of the boundary $\partial\Omega$ such that their outer normal is $\mathbf{n} = (\pm1, 0)$. Also this step is done to avoid some technical computations later on.

4.1. Definition and basic properties of $\rho(1,\mathbf{b})$-proper sets.

We start by giving the following definition.

Definition 4.1 (Proper sets)**.** An open, bounded set $\Omega \subset \mathbb{R}^{d+1}$ is called $\rho(1,\mathbf{b})$-*proper* if:

(1) $\partial\Omega$ has finite $\mathcal{H}^d$-measure and it can be written as

$$\partial\Omega = \bigcup_{i\in\mathbb{N}} U_i \cup N,$$

where N is a closed set with $\mathcal{H}^d(N) = 0$ and $\{U_i\}_{i\in\mathbb{N}}$ are countably many C^1-hypersurfaces such that the following holds: for every $(t,x) \in U_i$, there exists a ball $B^{d+1}_r(t,x)$ such that $\partial\Omega \cap B^{d+1}_r(t,x) = U_i$;

(2) if the functions $\phi^{\delta,\pm}$ are given by

$$\phi^{\delta,+}(t,x) := \max\left\{1 - \frac{\operatorname{dist}((t,x),\Omega)}{\delta}, 0\right\}, \qquad \phi^{\delta,-}(t,x) := \min\left\{\frac{\operatorname{dist}((t,x),\mathbb{R}^{d+1}\setminus\Omega)}{\delta}, 1\right\}, \tag{4.1}$$

then

$$\lim_{\delta\searrow 0} |\rho(1,\mathbf{b})\cdot\nabla\phi^{\delta,\pm}|\,\mathcal{L}^{d+1} = |\rho(1,\mathbf{b})\cdot\mathbf{n}|\,\mathcal{H}^d\llcorner_{\partial\Omega}, \qquad w^\star\text{-}\mathscr{M}_b(\mathbb{R}^{d+1}).$$

In the following we will write *proper* instead of $\rho(1,\mathbf{b})$-proper when there is no ambiguity about the vector field.

Proposition 4.2. *Proper sets enjoy the following properties:*

(1) the Lebesgue value $\rho(1,\mathbf{b})\cdot\mathbf{n}\llcorner_{\partial\Omega}$ belongs to $L^1(\mathcal{H}^d\llcorner_{\partial\Omega})$;
(2) it holds

$$\lim_{\delta\searrow 0} \rho(1,\mathbf{b})\cdot\nabla\phi^{\delta,\pm}\mathcal{L}^{d+1} = \rho(1,\mathbf{b})\cdot\mathbf{n}\,\mathcal{H}^d\llcorner_{\partial\Omega}, \qquad w^\star\text{-}\mathscr{M}_b(\mathbb{R}^{d+1});$$

(3) $|\mu|(\partial\Omega) = 0$, where $\mu = \operatorname{div}_{t,x}(\rho(1,\mathbf{b}))$.

Proof. Point (1) follows from the well known fact that weakly convergent sequences are uniformly bounded.

To prove Point (2), let ξ^+ be a weak limit (up to subsequences) of the sequence $\rho(1,\mathbf{b})\cdot\nabla\phi^{\delta,+}\mathcal{L}^{d+1}$ and notice that, due to the weak l.s.c. of the norm, it holds

$$|\xi^+| \le |\rho(1,\mathbf{b})\cdot\mathbf{n}|\mathcal{H}^d\llcorner_{\partial\Omega}.$$

For notational convenience we will write $\xi^+\mathcal{H}^d\llcorner_{\partial\Omega}$. It is thus enough to prove the statement locally inside each set U_i for a fixed i: in particular, since the definition is invariant under C^1-diffeomorphisms as it can be easily checked, we can think Ω to be locally the set $\{s<0\}$ in some coordinate system $(s,y)\in\mathbb{R}\times\mathbb{R}^d$. For $\mathbf{a}\in\mathbb{R}$, $m\in\mathbb{N}$ set

$$E^m_{\mathbf{a}} = \Big\{y\in U_i : \big|\rho(1,\mathbf{b}) - \mathbf{a}\big| < 2^{-m}\Big\}.$$

Using the fact that $\mathcal{L}^d$-a.e. point $y\in E^m_{\mathbf{a}}$ is a Lebesgue for $\rho(1,\mathbf{b})$ w.r.t. the measure $\mathcal{L}^{d+1}$, for every ε we can find $\bar r>0$ and a compact subset $K^m_{\mathbf{a}}\subset E^m_{\mathbf{a}}$ such that $\mathcal{L}^d(E^m_{\mathbf{a}}\setminus K^m_{\mathbf{a}})<\varepsilon$ and for every $y\in K^m_{\mathbf{a}}$, $0<r<\bar r$ it holds

$$\frac{1}{r}\int_0^r \frac{1}{r^d}\int_{B^d_r(y)} \big|\rho(1,\mathbf{b})(y',s) - \mathbf{a}\big|\,dy'ds < (1+\varepsilon)2^{-m}.$$

Now, by Besicovitch' Theorem [AFP00, Theorem 2.17], we cover $K_{\mathbf{a}}$ with finitely many closed balls $B^d_r(y_j)$, $j=1,\dots,N_r$, of radius $r<\bar r$ such that

$$N_r r^d \le C_d\mathcal{L}^d(K^m_{\mathbf{a}}).$$

Then, since $\nabla\phi^{\delta,+}\simeq(1,0)$ by the C^1-regularity of the boundary, we have that

$$\begin{aligned}
&\int_{U_i\times\mathbb{R}} \big|\big(\rho(1,\mathbf{b})(s,y) - \mathbf{a}\big)\cdot\nabla\phi^{\delta,+}(s,y)\big|\,dyds \\
&\qquad\le \mathcal{O}(1)\sum_{i=1}^{N_r}\frac{1}{r}\int_0^r\int_{B^d_r(y_j)}\big|\rho(1,\mathbf{b})(s,y) - \mathbf{a}\big|\,dyds \\
&\qquad\quad + \frac{1}{r}\int_0^r\int_{U_i\setminus\bigcup_j B^d_r(y_j)}\big|\big(\rho(1,\mathbf{b})(s,y) - \mathbf{a}\big)\cdot\nabla\phi^{\delta,+}(s,y)\big|\,dyds \\
&\qquad\le \mathcal{O}(2^{-m})N_r r^d + \frac{\mathcal{O}(1)}{r}\int_0^r\int_{U_i\setminus K^m_{\mathbf{a}}}\Big[\big|\rho(1,\mathbf{b})(s,y)\cdot\nabla\phi^{\delta,+}(s,y)\big| + |\mathbf{a}|\Big]\,dyds \\
&\qquad\le \mathcal{O}(2^{-m})\mathcal{L}^d(K^m_{\mathbf{a}}) + \frac{\mathcal{O}(1)}{r}\int_0^r\int_{U_i\setminus K^m_{\mathbf{a}}}\Big[\big|\rho(1,\mathbf{b})(s,y)\cdot\nabla\phi^{\delta,+}(s,y)\big| + |\mathbf{a}|\Big]\,dyds.
\end{aligned}$$

Passing to the limit in r one concludes that for a test function ψ whose support is in $U_i \times (-c, c)$ with $c < \bar r$ it holds

$$\begin{aligned}
\bigg| \int_{s=0} \psi(0,y)\,\xi^+(y)\,\mathcal{L}^d(dy) - \int_{s=0} \psi(0,y)\,\mathbf{a}\cdot\mathbf{n}\,\mathcal{L}^d \bigg| \\
&\le \|\psi\|_\infty \liminf_{\delta\searrow 0} \int_{U_i\times\mathbb{R}} \Big| \big(\rho(1,\mathbf{b})(s,y) - \mathbf{a}\big)\cdot \nabla\phi^{\delta,+}(s,y) \Big|\, dyds \\
&\le \mathcal{O}(2^{-m}\|\psi\|_\infty)\mathcal{L}^d(K^m_{\mathbf{a}}) + \|\psi\|_\infty \int_{U_i\setminus K^m_{\mathbf{a}}} \Big[\big|\rho(1,\mathbf{b})(0,y)\cdot\mathbf{n}(y)\big| + |\mathbf{a}| \Big]\, dy.
\end{aligned}$$

By considering a sequence of $\psi \le 1$ converging to $\mathbb{1}_{K^m_{\mathbf{a}}}$ and whose support is a subset of V open, $V \supset K^m_{\mathbf{a}}$, the above inequality gives that for every open set

$$\bigg| \int_{K^m_{\mathbf{a}}} \big(\xi^+(y) - \mathbf{a}\cdot\mathbf{n}\big)\,\mathcal{L}^d(dy) \bigg| \le \mathcal{O}(2^{-m})\mathcal{L}^d(K^m_{\mathbf{a}}) + \int_{V\setminus K^m_{\mathbf{a}}} \Big[\big|\rho(1,\mathbf{b})(0,y)\cdot\mathbf{n}(y)\big| + |\mathbf{a}| \Big]\, dy.$$

Letting now $V \searrow K^m_{\mathbf{a}}$ and then $\varepsilon \to 0$, we obtain that

$$\big|\xi^+(y) - \rho(1,\mathbf{b})(0,y)\cdot\mathbf{n}\big| \le C_d 2^{1-m} \qquad \mathcal{H}^d\text{-a.e. on } K^m_{\mathbf{a}}.$$

In particular the same holds in $E^m_{\mathbf{a}}$, by inner regularity of Radon measures. In particular by letting $m \to \infty$ we conclude that $\xi^+(y) = \rho(1,\mathbf{b})\cdot\mathbf{n}(y)$ for $\mathcal{H}^d$-a.e. $y \in V$.
The proof for the other case is completely similar.

The last point is a consequence of the second, as it holds

$$\xi^+ = \xi^- = \rho(1,\mathbf{b})\cdot\mathbf{n}\mathcal{H}^d \llcorner_{\partial\Omega},$$

thus $|\mu|(\partial\Omega) = 0$, where ξ^- is the weak limit of the sequence $\rho(1,\mathbf{b})\cdot\nabla\phi^{\delta,-}$ as $\delta \searrow 0$. □

Remark 4.3. It is possible to provide a more general class of proper sets as follows: let $f : \mathbb{R}^{d+1} \to \mathbb{R}$ be a Lipschitz function whose level sets $E_h = f^{-1}((h,+\infty))$ are compact: assume that there exists a closed set $N \subset \mathbb{R}^{d+1}$, with $\mathcal{H}^d(N) = 0$, such that $f \in C^1(\mathbb{R}^{d+1}\setminus N)$ and $\nabla f \neq 0$ in $\mathbb{R}^{d+1}\setminus N$. By the Coarea Formula (Theorem 3.7) and the local invertibility of C^1-functions outside critical points, it follows that for $\mathcal{L}^1$-a.e. h the set E_h satisfies Point (1) and Point (**??**).

Define now the functions

$$\phi^{\delta,+}_h = \bigg[1 - \frac{1}{\delta}[h-f]^+\bigg]^+, \qquad \phi^{\delta,-}_h = \min\bigg\{1, \frac{1}{\delta}[f-h]^+\bigg\}.$$

Condition (2) of Definition 4.1 is then replaced by

$$\lim_{\delta\searrow 0} \big|\rho(1,\mathbf{b})\cdot\nabla\phi^{\delta,\pm}_h\big|\,\mathcal{L}^{d+1} = \bigg|\rho(1,\mathbf{b})\cdot\frac{\nabla_{t,x}f}{|\nabla_{t,x}f|}\bigg|\,\mathcal{H}^d\llcorner_{\partial E_h}. \tag{4.2}$$

in the weak*-convergence of measures.

Being the map

$$\mathbb{R} \ni h \mapsto \bigg|\rho(1,\mathbf{b})\cdot\frac{\nabla_{t,x}f}{|\nabla_{t,x}f|}\bigg|\,\mathcal{H}^d\llcorner_{\partial E_h} \in \mathcal{M}_b(\mathbb{R}^{d+1})$$

an integrable map, it follows that by Lusin's Theorem that (4.2) holds for $\mathcal{L}^1$-a.e. h.

Proof. We give a proof of the above statement since in our case it is quite straightforward. The general case can be obtained by applying [Fre06, Theorem 4.18J].

Consider the finite measure m on $\mathbb{R}$ defined by

$$m(dh) = \bigg(\int_{\partial E_h} \bigg|\rho(1,\mathbf{b})\cdot\frac{\nabla_{t,x}f}{|\nabla_{t,x}f|}\bigg|\,\mathcal{H}^d \bigg)\,\mathcal{L}^1(dh).$$

If $\psi_n \in C^0(\mathbb{R}^{d+1},\mathbb{R})$, $n \in \mathbb{N}$, is a dense sequence of test functions, then by the standard Lusin's Theorem in $\mathbb{R}$ we obtain that up to an open set N_n such that $m(N_n) < \varepsilon 2^{-n}$ the function

$$h \mapsto d_{\psi_n}(h) := \int_{\partial E_h} \bigg|\rho(1,\mathbf{b})\cdot\frac{\nabla_{t,x}f}{|\nabla_{t,x}f|}\bigg|\psi_n\,\mathcal{H}^d$$

is continuous. By closure of the set $\{\psi_n\}_n$, it follows that

$$h \mapsto d_\psi(h) := \int_{\partial E_h} \left| \rho(1,\mathbf{b}) \cdot \frac{\nabla_{t,x} f}{|\nabla_{t,x} f|} \right| \psi \, \mathcal{H}^d$$

is continuous in $\mathbb{R} \setminus \cup_n N_n$, and being

$$d_\psi \, \mathcal{L}^1 \leq \|\psi\|_\infty \, m$$

it follows that every Lebesgue density point of $\mathbb{R} \setminus \cup_n N_n$ w.r.t. the measure m is a Lebesgue point of d_ψ. Being $\mathcal{L}^1(\cup_n N_n) < \varepsilon$, the conclusion follows. □

Remark 4.4. By means of the notion of trace introduced in following Section 5, it is also possible to refine the definition of proper sets as follows:

Definition 4.5 (Inner proper sets)**.** An open, bounded set $\Omega \subset \mathbb{R}^{d+1}$ is called $\rho(1,\mathbf{b})$-*inner proper* if:

(1) $\partial\Omega$ has finite $\mathcal{H}^d$-measure and it can be written as

$$\partial\Omega = \bigcup_{i \in \mathbb{N}} U_i \cup N,$$

where N is a closed set with $\mathcal{H}^d(N) = 0$ and $\{U_i\}_{i\in\mathbb{N}}$ are countably many C^1-hypersurfaces such that the following holds: for every $(t,x) \in U_i$, there exists a ball $B^{d+1}_r(t,x)$ such that $\partial\Omega \cap B^{d+1}_r(t,x) = U_i$;

(2) the distributional inner normal trace $\mathsf{Tr}^{\mathrm{in}}(\rho(1,\mathbf{b}),\Omega)\cdot\mathbf{n}$ of the vector field $\rho(1,\mathbf{b})$ is a measure and satisfies

$$\mathsf{Tr}^{\mathrm{in}}\big(\rho(1,\mathbf{b}),\Omega\big)\cdot\mathbf{n} \ll \mathcal{H}^d \llcorner_{\partial\Omega}.$$

As in the next section, in this case we will denote the trace as

$$\mathsf{Tr}^{\mathrm{in}}\big(\rho(1,\mathbf{b}),\Omega\big)\cdot\mathbf{n}\, \mathcal{H}^d \llcorner_{\partial\Omega},$$

i.e. as a function in $L^1(\mathcal{H}^d\llcorner_{\partial\Omega})$;

(3) if

$$\phi^{\delta,-}(x) := \min\left\{ \frac{\operatorname{dist}(x, \mathbb{R}^{d+1}\setminus\Omega)}{\delta}, 1 \right\}, \tag{4.3}$$

then

$$\lim_{\delta\searrow 0} \big|\rho(1,\mathbf{b})\cdot\nabla\phi^{\delta,-}\big|\, \mathcal{L}^{d+1} = \big|\mathsf{Tr}^{\mathrm{in}}\big(\rho(1,\mathbf{b}),\Omega\big)\cdot\mathbf{n}\big|\, \mathcal{H}^d\llcorner_{\partial\Omega}, \qquad w^\star\text{-}\mathscr{M}_b(\mathbb{R}^{d+1}).$$

A similar definition for $\rho(1,\mathbf{b})$-*outer proper*, i.e. $\mathbb{R}^{d+1}\setminus\operatorname{clos}\Omega$ is $\rho(1,\mathbf{b})$-inner proper. If the outer and inner normal traces coincide and the boundary $\partial\Omega$ is made of Lebesgue points, then Ω is $\rho(1,\mathbf{b})$-proper: it is fairly easy to construct an example showing that Condition (**??**) of Definition 4.1 is not implied by being inner and outer regular.

Remark 4.6. One can extend the definition of proper sets to sets with Lipschitz boundary (i.e. locally graph of Lipschitz functions), being the relevant quantities (i.e. Conditions (**??**), (2) of Definition 4.1) still meaningful. Also the use of Lebesgue points on $\partial\Omega$ is not needed, one can just take $\rho(1,\mathbf{b})\cdot\mathbf{n}$ as its inner/outer trace.

The $\rho(1,\mathbf{b})$-proper sets are used for testing purposes, so requiring additional regularity is not a problem when we can prove that there are sufficiently many of these sets.

To show that there are sufficiently many proper sets and to construct perturbations of these sets which are particularly suited for the study of the distribution $\rho(1,\mathbf{b})\llcorner_\Omega$, we consider the following family of sets. As usual we assume that $\mathbf{b}$ is a Borel function, hence defined everywhere.

Definition 4.7. For every fixed $(t,x)\in\mathbb{R}^{d+1}$ and $r, L > 0$, the *cylinder of center* (t,x) *and sizes* r, L (see Figure)is defined by

$$\mathrm{Cyl}^{r,L}_{t,x} = \Big\{(\tau,y) : |\tau - t| \leq Lr, \big|y - x - \mathbf{b}(t,x)(\tau - t)\big| < r\Big\}.$$

We now show that almost all balls and cylinders are proper sets: indeed, we have the following

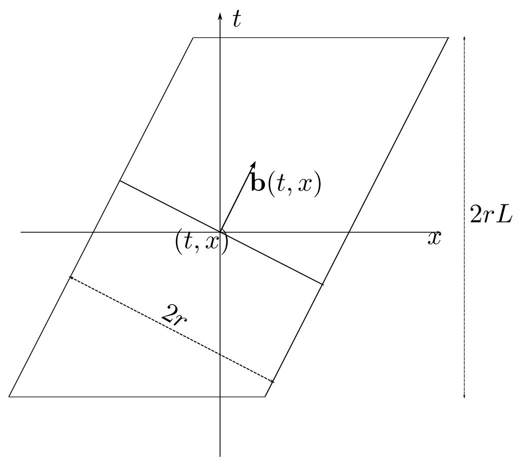


Figure 4. The cylinder $\mathrm{Cyl}^{r,L}_{t,x}$.

Lemma 4.8. *For every (t,x) consider the family of balls $\{B^{d+1}_r(t,x)\}_{r>0}$ and the family of cylinders $\{\mathrm{Cyl}^{r,L}_{t,x}\}_{r>0}$ with $L>0$ fixed. Then for $\mathcal{L}^1$-a.e. $r>0$ the ball $B^{d+1}_r(t,x)$ and the cylinder $\mathrm{Cyl}^{r,L}_{t,x}$ are proper sets.*

Proof. The statement is a consequence of Remark 4.3, respectively using the Lipschitz functions

$$(\tau,y)\mapsto \big|(\tau,y)-(t,x)\big|, \quad (\tau,y)\mapsto \max\big\{\big|y-x-\mathbf{b}(t,x)(\tau-t)\big|, |\tau-t|/L\big\}. \qquad \square$$

Proposition 4.9. *If Ω_1,Ω_2 are proper sets with*

$$\mathcal{H}^d\Big(\mathrm{Fr}\left(\partial\Omega_1\cap\partial\Omega_2, \partial\Omega_1\cup\partial\Omega_2\right)\Big)=0, \tag{4.4}$$

then $\Omega:=\Omega_1\cup\Omega_2$ is proper.

Proof. Clearly, the set Ω is piecewise C^1 and the set of Lebesgue points of $\rho(1,\mathbf{b})$ has full measure. It remains to prove Condition (2) of Definition 4.1. We will study only $\phi^{\delta,+}$.

If $\phi^{\delta,+}_i$ is the function given by the first formula of (4.3) for Ω_i, with $i=1,2$, observe that

$$\phi^{\delta,+}=\max\left\{\phi^{\delta,+}_1,\phi^{\delta,+}_2\right\}$$

and we write for any continuous function ψ

$$\begin{aligned}
&\int \big|\rho(1,\mathbf{b})\cdot\nabla\phi^{\delta,+}\big|\psi\,\mathcal{L}^{d+1}\\
&\quad=\left[\int_{A_1}+\int_{A_2}+\int_{A_3}\right]\big|\rho(1,\mathbf{b})\cdot\nabla\phi^{\delta,+}\big|\psi\,\mathcal{L}^{d+1}\\
&\quad=\int_{A_1}|\rho(1,\mathbf{b})\cdot\nabla\phi^{\delta,+}_1|\psi\,\mathcal{L}^{d+1}+\int_{A_2}|\rho(1,\mathbf{b})\cdot\nabla\phi^{\delta,+}_2|\psi\,\mathcal{L}^{d+1}+\int_{A_3}|\rho(1,\mathbf{b})\cdot\nabla\phi^{\delta,+}|\psi\,\mathcal{L}^{d+1}
\end{aligned}$$

where

$$\begin{aligned}
A_1&=\Big\{(t,x):\mathrm{dist}\left((t,x),\Omega_1\right)<\mathrm{dist}\left((t,x),\Omega_2\right)\Big\},\\
A_2&=\Big\{(t,x):\mathrm{dist}\left((t,x),\Omega_2\right)<\mathrm{dist}\left((t,x),\Omega_1\right)\Big\},\\
A_3&=\Big\{(t,x):\mathrm{dist}\left((t,x),\Omega_1\right)=\mathrm{dist}\left((t,x),\Omega_2\right)\Big\}.
\end{aligned}$$

We prove that

$$\int_{A_1}\big|\rho(1,\mathbf{b})\cdot\nabla\phi^{\delta,+}_1\big|\psi\,\mathcal{L}^{d+1}\to\int_{\partial\Omega_1\setminus\mathrm{clos}\,\Omega_2}\big|\rho(1,\mathbf{b})\cdot\mathbf{n}\big|\psi\,\mathcal{H}^d. \tag{4.5}$$

Consider the set $\mathrm{int}(A_1,\partial\Omega)$ which is relatively open by definition, so that by l.s.c. of the weak convergence on open sets we deduce

$$\big|\rho(1,\mathbf{b})\cdot\mathbf{n}\big|\,\mathcal{H}^d\llcorner_{\mathrm{int}(A_1;\partial\Omega)}\le\liminf_{\delta\to0}\big|\rho(1,\mathbf{b})\cdot\nabla\phi^{\delta,+}_1\big|\,\mathcal{L}^{d+1}\llcorner_{A_1}.$$

On the other hand,

$$\mathrm{clos}(A_1, \partial\Omega) \subset \mathrm{int}(A_1, \partial\Omega) \cup \mathrm{Fr}(A_3, \partial\Omega)$$

and

$$\mathrm{Fr}(A_3, \partial\Omega) = \mathrm{Fr}(\partial\Omega_1 \cap \partial\Omega_2, \partial\Omega) = \mathrm{Fr}(\partial\Omega_1 \cap \partial\Omega_2, \partial\Omega_1 \cup \partial\Omega_2).$$

Being the latter sets $\mathcal{H}^d$-negligible (by assumption) and using the u.s.c. of the weak convergence on closed set ($\mathrm{clos}(A_1, \partial\Omega)$ in this case), we get

$$\begin{aligned}\limsup_{\delta\to 0} \big|\rho(1,\mathbf{b}) \cdot \nabla\phi_1^{\delta,+}\big| \mathcal{L}^{d+1} \llcorner_{A_1} &\leq \big|\rho(1,\mathbf{b}) \cdot \mathbf{n}\big| \mathcal{H}^d \llcorner_{\mathrm{clos}(A_1,\partial\Omega)} \\ &\leq \big|\rho(1,\mathbf{b}) \cdot \mathbf{n}\big| \mathcal{H}^d \llcorner_{\mathrm{int}(A_1,\partial\Omega)\cup\mathrm{Fr}(A_3,\partial\Omega)} \\ &= \big|\rho(1,\mathbf{b}) \cdot \mathbf{n}\big| \mathcal{H}^d \llcorner_{\mathrm{int}(A_1,\partial\Omega)}.\end{aligned}$$

This gives (4.5).
The proof for A_2 is analogous, i.e.

$$\int_{A_2} \big|\rho(1,\mathbf{b}) \cdot \nabla\phi_2^{\delta,+}\big| \psi\, \mathcal{L}^{d+1} \to \int_{\partial\Omega_2 \setminus \mathrm{clos}\,\Omega_1} \big|\rho(1,\mathbf{b}) \cdot \mathbf{n}\big| \psi\, \mathcal{H}^d. \tag{4.6}$$

Finally it holds

$$\phi^{\delta,+}\llcorner_{\mathrm{int}\,A_3} = \phi_1^{\delta,+}\llcorner_{\mathrm{int}\,A_3} = \phi_2^{\delta,+}\llcorner_{\mathrm{int}\,A_3},$$

and then in a completely similar way for A_3

$$\int_{\mathrm{int}\,A_3} \big|\rho(1,\mathbf{b}) \cdot \nabla\phi_1^{\delta,+}\big| \psi\, \mathcal{L}^{d+1} \to \int_{\partial\Omega_3 \setminus \mathrm{Fr}(\partial\Omega_1\cap\partial\Omega_2,\partial\Omega)} \big|\rho(1,\mathbf{b}) \cdot \mathbf{n}\big| \psi\, \mathcal{H}^d. \tag{4.7}$$

Concerning the set of point on ∂A_3, it follows that for $\delta \ll 1$

$$\int_{\partial A_3} \big|\rho(1,\mathbf{b}) \cdot \nabla\phi^{\delta,+}\big| \psi\, \mathcal{L}^{d+1} \leq \int_O \big|\rho(1,\mathbf{b}) \cdot \nabla\phi_1^{\delta,+}\big| \psi \mathcal{L}^{d+1},$$

where O is an open neighborhood in $\mathbb{R}^{d+1}$ of $\mathrm{Fr}(\partial\Omega_1 \cap \partial\Omega_2, \partial\Omega)$ containing the support of ψ. Hence

$$\limsup_{\delta\to 0} \int_{\partial A_3} \big|\rho(1,\mathbf{b}) \cdot \nabla\phi^{\delta,+}\big| \psi\, \mathcal{L}^{d+1} \leq \int_{O\cap\partial\Omega} \big|\rho(1,\mathbf{b}) \cdot \mathbf{n}\big| |\psi|\, \mathcal{H}^d \leq \|\psi\|_\infty \int_{O\cap\partial\Omega} \big|\rho(1,\mathbf{b}) \cdot \mathbf{n}\big| \mathcal{H}^d,$$

and by the assumption on the $\mathcal{H}^d$-negligibility of $\mathrm{Fr}(\partial\Omega_1 \cap \partial\Omega_2, \partial\Omega)$ one obtains that this integral is arbitrarily small. Adding (4.5), (4.6) and (4.7) the conclusion follows. □

The above proposition allows to construct sufficiently many proper sets for our purposes, starting from Lemma 4.8.

Corollary 4.10. *The finite union of proper balls and proper cylinders is proper.*

Proof. Indeed their intersection has the property (4.4) by elementary geometry. □

4.2. **Perturbation of proper sets.**

Let $\Omega \subset \mathbb{R}^{d+1}$ be a $\rho(1,\mathbf{b})$-proper set.

Lemma 4.11. *For every $\varepsilon > 0$ there exist a compact set $K^\varepsilon \subset \partial\Omega \setminus N$ and $\alpha > 0$ with the following properties:*

(1) $\alpha^{-1} < \rho, |(1,\mathbf{b}) \cdot \mathbf{n}|$ and $\rho, |\mathbf{b}| < \alpha$ for $\mathcal{H}^d$-a.e. $(t,x) \in K^\varepsilon$;
(2) the remaining set has small normal trace, i.e.

$$\int_{\partial\Omega\setminus K^\varepsilon} \rho|(1,\mathbf{b}) \cdot \mathbf{n}|\, \mathcal{H}^d < \varepsilon.$$

Proof. It is enough to observe that

$$\lim_{\alpha\to+\infty} \int_{\partial\Omega\cap\{\alpha^{-1}<\rho,|(1,\mathbf{b})\cdot\mathbf{n}|\}\cap\{\rho,|\mathbf{b}|<\alpha\}} \rho(t,x)|(1,\mathbf{b}(t,x)) \cdot \mathbf{n}|\, \mathcal{H}^d(dtdx) = \int_{\partial\Omega} \rho|(1,\mathbf{b}) \cdot \mathbf{n}|\, \mathcal{H}^d,$$

since $\rho|(1,\mathbf{b}) \cdot \mathbf{n}|$ is an L^1-function w.r.t. $\mathcal{H}^d\llcorner_{\partial\Omega}$ and

$$\partial\Omega \subset \big\{\rho, |(1,\mathbf{b}) \cdot \mathbf{n}|, |\mathbf{b}| = 0\big\} \cup \bigcup_\alpha \big\{\alpha^{-1} < \rho, |(1,\mathbf{b}) \cdot \mathbf{n}|\big\} \cap \big\{\rho, |\mathbf{b}| < \alpha\big\}$$

being Borel functions. □

By simple geometric manipulation, it follows that for r sufficiently small and $L > 2\alpha^2$ ($L > \alpha^2$ would be enough for most of the theorems, but later we need some extra room) the cylinder

$$\mathrm{Cyl}^{r,L}_{t,x} = \Big\{(\tau, y) : |\tau - t| < Lr, \big|y - x - \mathbf{b}(t,x)(\tau - t)\big| < r\Big\}$$

has top and bottom faces contained one inside Ω and the other outside, for every point in $(t,x) \in K^\varepsilon$: more precisely, if $(1,\mathbf{b}) \cdot \mathbf{n} > 0$ then

$$\Big\{(t + Lr, y) : \big|y - x - L\mathbf{b}(t,x)r\big| < r\Big\} \subset \mathbb{R}^{d+1} \setminus \operatorname{clos}\Omega, \tag{4.8a}$$

$$\Big\{(t - Lr, y) : \big|y - x + L\mathbf{b}(t,x)r\big| < r\Big\} \subset \Omega. \tag{4.8b}$$

The opposite relations hold for $(1,\mathbf{b}) \cdot \mathbf{n} < 0$. Moreover, being $\partial\Omega$ of class C^1 in a neighborhood of $\cap K^\varepsilon$, we have

$$B^{d+1}_{r/L}(t,x) \cap \partial\Omega \subset \mathrm{Cyl}^{r,L}_{t,x} \cap \partial\Omega \subset B^{d+1}_{Lr}(t,x) \cap \partial\Omega, \tag{4.9a}$$

$$\mathcal{H}^d(\partial\mathrm{Cyl}^{r,L}_{t,x} \cap \partial\Omega) = 0, \tag{4.9b}$$

again by simple geometrical arguments.

We now recall the following elementary

Lemma 4.12. *If (t,x) is a Lebesgue point for $\rho(1,\mathbf{b})$, then for every $L > 0$ fixed it holds*

$$\lim_{r\to 0} \frac{1}{r}\int_0^r \left[\frac{1}{r^d}\int_{\partial\mathrm{Cyl}^{r,L}_{t,x}} \big|\rho(1,\mathbf{b})(\tau,y) - \rho(1,\mathbf{b})(t,x)\big|\,\mathcal{H}^{d-1}(dy)d\tau\right] ds = 0.$$

Proof. We have, using Fubini's Theorem,

$$\begin{aligned}
&\frac{1}{r}\int_0^r \left[\frac{1}{r^d}\int_{\partial\mathrm{Cyl}^{r,L}_{t,x}} \big|\rho(1,\mathbf{b})(\tau,y) - \rho(1,\mathbf{b})(t,x)\big|\,\mathcal{H}^{d-1}(dy)d\tau\right] ds \\
&\quad = \frac{1}{r}\int_0^r \left[\frac{1}{r^d}\int_{t-Lr}^{t+Lr}\int_{\partial B^d_r(x - \mathbf{b}(t,x)(\tau - t))} \big|\rho(1,\mathbf{b})(\tau,y) - \rho(1,\mathbf{b})(t,x)\big|\mathcal{H}^{d-1}(dy)d\tau\right] ds \\
&\quad = \frac{1}{r^{d+1}}\int_{t-Lr}^{t+Lr}\int_{B^d_r(x - \mathbf{b}(t,x)(\tau - t))} \big|\rho(1,\mathbf{b})(\tau,y) - \rho(1,\mathbf{b})(t,x)\big| dy d\tau \\
&\quad \leq \frac{1}{r^{d+1}}\int_{B^d_{(1+L|\mathbf{b}|(t,x))r}(t,x)} \big|\rho(1,\mathbf{b})(\tau,y) - \rho(1,\mathbf{b})(t,x)\big| dy d\tau \\
&\quad = \omega_{d+1}(1 + L|\mathbf{b}|(t,x))^{d+1} \fint_{B^d_{(1+L|\mathbf{b}|(t,x))r}(t,x)} \big|\rho(1,\mathbf{b})(\tau,y) - \rho(1,\mathbf{b})(t,x)\big| dy d\tau \to 0,
\end{aligned}$$

since (t,x) is a Lebesgue point for $\rho(1,\mathbf{b})$. This implies the statement. $\square$

Using Lemma 4.8 and 4.12, we have that for every fixed $\varepsilon' > 0$, for any $(t,x) \in K^\varepsilon$ Lebesgue point for $\rho(1,\mathbf{b})$, we can choose the $r < \varepsilon'$ such that:

- $\mathrm{Cyl}^{r,L}_{t,x}$ is proper;
- it holds
$$\frac{1}{r^d}\int_{t-Lr}^{t+Lr}\int_{\partial B^d_r(x - \mathbf{b}(t,x)(\tau - t))} \big|\rho(1,\mathbf{b})(\tau,y) - \rho(1,\mathbf{b})(t,x)\big|\,\mathcal{H}^{d-1}(dy)d\tau < \varepsilon'; \tag{4.10}$$
- conditions (4.8) hold;
- $\mathrm{Cyl}^{r,L}_{t,x} \cap \partial\Omega$ is equivalent to a ball and its boundary is $\mathcal{H}^d$ negligible, i.e. (4.9) hold.

In the following we will call a cylinder satisfying the above condition *$\rho(1,\mathbf{b})$-proper (ε',Ω)-regular cylinder*, or for brevity *proper regular* whenever the vector field $\rho(1,\mathbf{b})$ and dependence on the ε' or Ω is clear from the context or not essential to the computation.

We can proceed further by observing that 0 is a Lebesgue density point for the set satisfying (4.10) for all $\varepsilon' > 0$. On the other hand, it is easy to see that the other three properties are verified $\mathcal{L}^1$-a.e. r. We state it in the following lemma.

Lemma 4.13. *If $(t,x) \in K^\varepsilon$ is a Lebesgue point for $\rho(1,\mathbf{b})$, the set of r such that $\mathrm{Cyl}^{r,L}_{t,x}$ satisfies the above condition has 0 as a Lebesgue point w.r.t. the measure $\mathcal{L}^1$:*

$$\lim_{r\searrow 0} \frac{1}{r}\mathcal{L}^1\Big(\Big\{r' \in (0,r) : \mathrm{Cyl}^{r',L}_{t,x} \text{ is proper } (\varepsilon',\Omega)\text{-regular}\Big\}\Big) = 1.$$

Thus we obtain the following extension of Lemma 4.11:

Lemma 4.14. *For every $\varepsilon' > 0$, there exists $\bar r > 0$ and a compact set $K^{\varepsilon,\varepsilon'}_{\bar r} \subset K^\varepsilon$ made of Lebesgue points of $\rho(1,\mathbf{b})$ such that*

(1) $\alpha^{-1} < \rho, |(1,\mathbf{b})\cdot\mathbf{n}|$ and $\rho, |\mathbf{b}| < \alpha$ for $\mathcal{H}^d$-a.e. $(t,x) \in K^{\varepsilon,\varepsilon'}_{\bar r}$;
(2) the remaining set has small normal trace,

$$\int_{\partial\Omega\setminus K^{\varepsilon,\varepsilon'}_{\bar r}} \rho|(1,\mathbf{b})\cdot\mathbf{n}|\,\mathcal{H}^d < 2\varepsilon,$$

and for every $(t,x) \in K^{\varepsilon,\varepsilon'}_{\bar r}$, $r' \le \bar r$ there exists a proper (ε',Ω)-regular cylinder $\mathrm{Cyl}^{r,L}_{t,x}$ with $r'/2 < r < r'$.

By (4.9) we get the next proposition.

Proposition 4.15. *For every $r' \le \bar r$, there exists a finite covering of $K^{\varepsilon,\varepsilon'}_{\bar r}$ with cylinders $\{\mathrm{Cyl}^{r_i,L}_{t_i,x_i}\}_{i=1}^{N_{r'}}$ with $L > 2\alpha^2$ and $r'/2 < r_i < r'$, such that*

- *they are all proper (ε',Ω)-regular,*
- *it holds*

$$N_{r'}(r')^d \le C_d L^d \mathcal{H}^d(K^{\varepsilon,\varepsilon'}_{\bar r})$$

and

$$\sum_{i=1}^{N_{r'}} \int_{t_i-Lr_i}^{t_i+Lr_i} \int_{\partial B^d_{r_i}(x_i-\mathbf{b}(t_i,x_i)(t-t_i))} \big|\rho(1,\mathbf{b})\cdot\mathbf{n}\big|\,\mathcal{H}^d \le (1+\alpha)C_d\varepsilon' L^d\mathcal{H}^d(K^{\varepsilon,\varepsilon'}_{\bar r}). \tag{4.11}$$

Proof. By Lemma 4.13 for every point of $K^{\varepsilon,\varepsilon'}_{\bar r}$, $r' \le \bar r$ we can find cylinders $\mathrm{Cyl}^{r,L}_{t,x}$ which are proper sets with $r'/2 < r < r'$ and by (4.9a) their intersection with $\partial\Omega$ is equivalent to balls (by the assumption $L > 2\alpha^2$), so that by Besicovitch Theorem [AFP00, Theorem 2.17] we can take a covering $\{\mathrm{Cyl}^{r_i,L}_{t_i,x_i}\}_{i=1}^{N_{r'}}$ satisfying

$$2^{-d}N_{r'}\left(\frac{r'}{L}\right)^d \le \sum_{i=1}^{N_{r'}} \mathcal{H}^d\big(\mathrm{Cyl}^{r_i,L}_{t_i,x_i} \cap \partial\Omega\big) \le C_d\mathcal{H}^d(K^{\varepsilon,\varepsilon'}_{\bar r}),$$

with C_d constant depending only on the dimension. The constant L^d is a consequence of (4.9a). The other claim follows from (4.10), because of the triangle inequality

$$\begin{aligned}
&\int_{t_i-Lr_i}^{t_i+Lr_i}\int_{\partial B^d_{r_i}(x-\mathbf{b}(t_i,x_i)(t-t_i))} \rho\big|(1,\mathbf{b})\cdot\mathbf{n}\big|\,\mathcal{H}^d \\
&\quad\le \int_{t_i-Lr_i}^{t_i+Lr_i}\int_{\partial B^d_{r_i}(x-\mathbf{b}(t_i,x_i)(t-t_i))} \rho(\tau,x)\big|\mathbf{b}(\tau,x)-\mathbf{b}(t_i,x_i)\big|\,\mathcal{H}^{d-1}(dx)d\tau \\
&\quad\le \int_{t_i-Lr_i}^{t_i+Lr_i}\int_{\partial B^d_{r_i}(x-\mathbf{b}(t_i,x_i)(t-t_i))} \big|\rho(\tau,x)\mathbf{b}(\tau,x)-\rho(t_i,x_i)\mathbf{b}(t,x)\big|\,\mathcal{H}^{d-1}(dx)d\tau \\
&\qquad + |\mathbf{b}(t_i,x_i)|\int_{t_i-Lr_i}^{t_i+Lr_i}\int_{\partial B^d_{r_i}(x-\mathbf{b}(t_i,x_i)(t-t_i))} \big|\rho(t,x)-\rho(t_i,x_i)\big|\,\mathcal{H}^{d-1}(dx)d\tau \\
&\quad\le (1+\alpha)\varepsilon' r_i^d.
\end{aligned}$$

□

We thus obtain the main result of this section.

Theorem 4.16. *For every $\varepsilon > 0$ there exists a proper set Ω^ε such that*

(1) $\Omega \subset \Omega^\varepsilon \subset \Omega + B^{d+1}_\varepsilon(0)$;

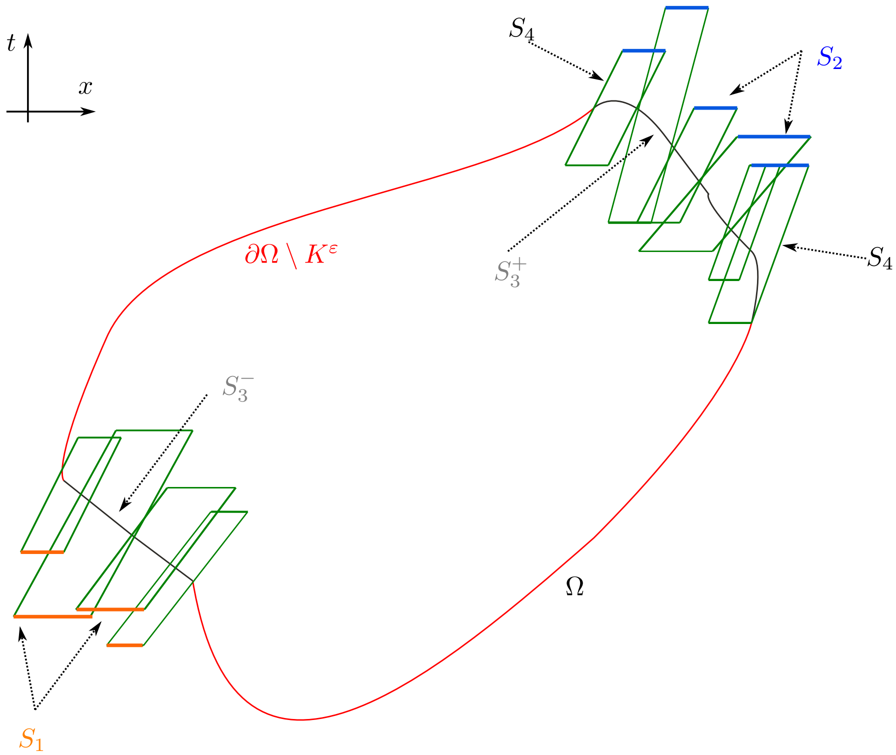


Figure 5. Perturbation of the proper set Ω constructed in Theorem 4.16.

(2) if

$$\partial\Omega_1^\varepsilon = \Big\{(t,x) \in \partial\Omega^\varepsilon : \mathbf{n} = (1,0) \text{ in a neigborhood of } (t,x)\Big\},$$

then $\partial\Omega_1^\varepsilon$ *is made of Lebesgue points of* $\rho(1,\mathbf{b})$ *and*

$$\left| \int_{\partial\Omega_1^\varepsilon} \rho\mathcal{H}^d - \int_{\partial\Omega} \rho[(1,\mathbf{b})\cdot\mathbf{n}]^+ \,\mathcal{H}^d \right| < \varepsilon;$$

(3) if

$$\partial\Omega_2^\varepsilon = \Big\{(t,x) \in \partial\Omega^\varepsilon : \mathbf{n} = (-1,0) \text{ in a neigborhood of } (t,x)\Big\},$$

then $\partial\Omega_2^\varepsilon$ *is made of Lebesgue points of* $\rho(1,\mathbf{b})$ *and*

$$\left| \int_{\partial\Omega_2^\varepsilon} \rho\mathcal{H}^d - \int_{\partial\Omega} \rho[(1,\mathbf{b})\cdot\mathbf{n}]^- \,\mathcal{H}^d \right| < \varepsilon.$$

Additionally to the fact that proper sets can be perturbed, the advantage of the perturbations considered here is that essentially all inflow and outflow of $\rho(1,\mathbf{b})$ are occurring on open sets which are contained in countably many time-flat hyperplanes (see 5). Due to the special form of the vector field, many computations occurring in the next sections are greatly simplified.

Proof. First we find a compact set $K^{\varepsilon/7}$ such that Properties (1), (2) of the statement of Lemma 4.11 hold for $\varepsilon/7$. By inner regularity of the measure $\mathcal{H}^d$, we can further find two disjoint compact sets $K^{\varepsilon/6,\pm}$ such that $K^{\varepsilon/6} := K^{\varepsilon/6,+} \cup K^{\varepsilon/6,-}$ satisfies again Lemma 4.11 but

$$(1,\mathbf{b})\cdot\mathbf{n}\llcorner_{K^{\varepsilon/6,\pm}} \gtrless 0.$$

Choose ε' such that

$$(1+\alpha)C_d\varepsilon'(2\alpha)^d\mathcal{H}^d(\partial\Omega) < \frac{\varepsilon}{3}.$$

We apply Lemma 4.14 in order to obtain a a compact set $K_{\bar r}^{\varepsilon/6,\varepsilon'} \subset K^{\varepsilon/6}$ such that

$$\int_{\partial\Omega \setminus K_{\bar r}^{\varepsilon/6,\varepsilon'}} \rho|(1,\mathbf{b})\cdot \mathbf{n}|\,\mathcal{H}^d < \frac{\varepsilon}{3}.$$

Next, by Proposition 4.15 with

$$r' < \frac{\operatorname{dist}(K^{\varepsilon/6,+}, K^{\varepsilon/6,-})}{2(1+2\alpha^2)} \quad \text{such that} \quad |\mu|\big((\Omega + B^{d+1}_{r'}(\mathbf{0}))\setminus\Omega\big) < \frac{\varepsilon}{3}, \tag{4.12}$$

we conclude that there exists a covering of $K_{\bar r}^{\varepsilon/6,\varepsilon'}$ with finitely many ε'-proper regular cylinders $\{\mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\}_{i=1}^{N'_r}$, with $r'/2 < r_i < r'$ such that (4.11) holds. By the choice (4.12) it follows that the covering of $K^{\varepsilon/6,+}\cap K_{\bar r}^{\varepsilon/6,\varepsilon'}$ and $K^{\varepsilon/6,-}\cap K_{\bar r}^{\varepsilon/6,\varepsilon'}$ are disjoint.

Define

$$\Omega^\varepsilon := \Omega \cup \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}.$$

By Proposition 4.9 and Corollary 4.10 the set Ω^ε is proper and Point (1) is clearly satisfied.

To prove Point (2), partition the boundary of $\Omega^\varepsilon\setminus\Omega$ as

$$\begin{aligned}
\partial(\Omega^\varepsilon\setminus\Omega) = {} & \bigg[\partial\Omega^\varepsilon \cap \bigcup_{(t_i,x_i)\in K^{\varepsilon/6,+}} \Big\{(t_i + 2\alpha^2 L r_i, y) : \big|y - x_i - 2\alpha^2\mathbf{b}(t_i,x_i)\big| < r_i\Big\}\bigg] \\
& \cup \bigg[\partial\Omega^\varepsilon \cap \bigcup_{(t_i,x_i)\in K^{\varepsilon/6,-}} \Big\{(t_i - 2\alpha^2 L r_i, y) : \big|y - x_i + 2\alpha^2\mathbf{b}(t_i,x_i)\big| < r_i\Big\}\bigg] \\
& \cup \bigg[\partial\Omega \cap \bigcup_{(t_i,x_i)\in K^{\varepsilon/6,+}} \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\bigg] \cup \bigg[\partial\Omega \cap \bigcup_{(t_i,x_i)\in K^{\varepsilon/6,-}} \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\bigg] \cup S_4 \\
= {} & S_1 \cup S_2 \cup S_3^+ \cup S_3^- \cup S_4.
\end{aligned} \tag{4.13}$$

The set S_4 satisfies

$$S_4 \subset \bigcup_i \partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} := \bigcup_i \Big\{(\tau,y) : |\tau - t_i| \le 2\alpha^2 r_i, \big|y - x_i - \mathbf{b}(t_i,x_i)(\tau - t)\big| = r_i\Big\},$$

so that from (4.11)

$$\int_{S_4} \rho\big|(1,\mathbf{b})\cdot\mathbf{n}\big|\,\mathcal{H}^d \le (1+\alpha)C_d\varepsilon' L^d \mathcal{H}^d(\partial\Omega) < \frac{\varepsilon}{3}, \tag{4.14}$$

by the choice of ε'.

The balance of the equation $\operatorname{div}_{t,x}(\rho(1,\mathbf{b})) = \mu$ for the covering of $K_{\bar r}^{\varepsilon/6,\varepsilon'}\cap K^{\varepsilon/6,+}$ and the continuity property (4.12) give

$$\bigg|\int_{S_1} \rho\,\mathcal{H}^d - \int_{S_3^+} \rho[(1,\mathbf{b})\cdot\mathbf{n}]\,\mathcal{H}^d\bigg| \le \int_{S_4} \rho|(1,\mathbf{b})\cdot\mathbf{n}|\,\mathcal{H}^d + |\mu|(\Omega^\varepsilon\setminus\Omega) < \frac{2\varepsilon}{3}$$

and, from the properties of $K_{\bar r}^{\varepsilon/6,\varepsilon'}$, we eventually get

$$\bigg|\int_{S_3^+} \rho(1,\mathbf{b})\cdot\mathbf{n}\,\mathcal{H}^d - \int \rho[(1,\mathbf{b})\cdot\mathbf{n}]^+\,\mathcal{H}^d\bigg| \le \int_{\partial\Omega\setminus K_{\bar r}^{\varepsilon/6,\varepsilon'}} \rho|(1,\mathbf{b})\cdot\mathbf{n}|\,\mathcal{H}^d < \frac{\varepsilon}{3}.$$

This concludes the proof of Point (2) because $S_1 \subset \partial\Omega_1^\varepsilon$.

The proof of Property (3) is similar and it is omitted. □

5. Lagrangian representations and flow traces

In this section we study the effect of the functional operation

$$\rho(1,\mathbf{b})\,\mathcal{L}^{d+1} \mapsto \rho(1,\mathbf{b})\,\mathcal{L}^{d+1}\llcorner_\Omega,$$

where $\Omega\subset\mathbb{R}^{d+1}$ is an open set and

$$\operatorname{div}_{t,x}(\rho(1,\mathbf{b})) = \mu \in \mathscr{M}_b(I\times\mathbb{R}^d),$$

from the point of view of a Lagrangian representation η. We will show that there is a strong relation between the map $\gamma \mapsto \gamma\llcorner_{(\mathrm{id},\gamma)^{-1}(\Omega)}$, the trace operator and the construction of a Lagrangian representation for $\rho(1,\mathbf{b})\llcorner\Omega$ starting from a Lagrangian representation η for $\rho(1,\mathbf{b})$.

In this section we first show how one can use η to represent the trace of $\rho(1,\mathbf{b})$ as a (possibly non-absolutely convergent) sum of measures. The structure of the series is strongly related to the fact that $(\mathtt{R})_\sharp\eta$ is a Lagrangian representation for $\rho(1,\mathbf{b})\,\mathcal{L}^{d+1}\llcorner\Omega$: indeed, as shown in the second part of this section, this turns out to be true for a generic Lipschitz set if $\mathbf{b} \in L^1_t(\mathrm{BD}_x)_{\mathrm{loc}}$ and $\rho \in L^\infty_{\mathrm{loc}}$ (see Proposition 5.12). This will be proved using a chain rule formula for traces of BD functions (proved in [ACM05]).

5.1. **Flow traces.** The first step is to show that, using Lagrangian representations, it is possible to represent the normal trace over a generic closed set of a vector field $B = \rho(1,\mathbf{b}) \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1},\mathbb{R}^{d+1})$ with measure divergence as a (non absolutely convergent) sum of signed measures. In the case of a compact set $\Omega \subset \mathbb{R}^{d+1}$ with Lipschitz boundary and when $\mathbf{b} \in L^1_t(\mathrm{BD}_x)_{\mathrm{loc}}$, $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{d+1})$ the series turns out to be strongly convergent and thus gives back the usual definition of trace as a measure (absolutely continuous w.r.t $\mathcal{H}^d\llcorner\partial\Omega$). For general measure divergence vector fields, the same conclusion can be obtained when the set Ω is $\rho(1,\mathbf{b})$-proper, and it will be addressed in the next section.

We start by recalling some well known definitions.

5.1.1. *Definition of normal traces.* Let $\Omega \subset \mathbb{R}^{d+1}$ be an open set and let $B\colon \Omega \to \mathbb{R}^{d+1}$ be a locally integrable vector field with measure divergence, i.e.

$$B \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1},\mathbb{R}^{d+1}), \qquad \mathrm{div}_{t,x} B \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^{d+1}).$$

Definition 5.1. The *inner normal trace* of B over $\partial\Omega$ is the distribution denoted by $\mathsf{Tr}^{\mathrm{in}}(B,\Omega)\cdot\mathbf{n}$ and defined by

$$\big\langle \mathsf{Tr}^{\mathrm{in}}(B,\Omega)\cdot\mathbf{n},\psi\big\rangle := \int_\Omega \psi(t,x)\,(\mathrm{div}\,B)(dt,dx) + \int_\Omega B\cdot\nabla_{t,x}\psi(t,x)\,\mathcal{L}^{d+1}(dt,dx)$$

for every compactly supported smooth test function $\psi \in C^\infty_c(\mathbb{R}^{d+1})$. Similarly, we define the *outer normal trace* by

$$\mathsf{Tr}^{\mathrm{out}}(B,\Omega)\cdot\mathbf{n} := -\,\mathsf{Tr}^{\mathrm{in}}(B,\mathbb{R}^{d+1}\setminus\mathrm{clos}\,\Omega)\cdot\mathbf{n}. \tag{5.1}$$

Notice that

$$\big\langle \mathsf{Tr}^{\mathrm{out}}(B,\Omega)\cdot\mathbf{n},\psi\big\rangle - \big\langle \mathsf{Tr}^{\mathrm{in}}(B,\Omega)\cdot\mathbf{n},\psi\big\rangle = \int_{\partial\Omega}\psi\,(\mathrm{div}\,B) + \int_{\partial\Omega} B\cdot\nabla\psi\,\mathcal{L}^{d+1}.$$

In particular they coincide if $\partial\Omega$ is negligible w.r.t. both $\mathcal{L}^{d+1}$ and $\mathrm{div}\,B$.

Remark 5.2. Observe that in general $\mathbf{n}$ is not well defined without further assumptions on the set Ω: we use it only to keep the notation similar to the smooth case, where the value of B on $\partial\Omega$ is defined. Later on we will show that in the case of a proper set it coincides with the unit outer normal, and $\mathsf{Tr}^{\mathrm{in}/\mathrm{out}}(\rho(1,\mathbf{b}),\Omega)$ will be the Lebesgue value of the vector field on $\partial\Omega$, both defined $\mathcal{H}^d$-a.e..

5.1.2. *Traces for regular sets and vector fields.* If the domain Ω is sufficiently regular and if $B \in L^\infty(\mathbb{R}^{d+1})$ or $B = \rho B'$, with $\rho \in L^\infty(\mathbb{R}^{d+1})$ and $B' \in \mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^{d+1})$, there are well known results that allows to characterize the trace. We list here the main ones and we refer for more details to [Anz83] and [DL07, Chapter 7].

Proposition 5.3. *If Ω is of class C^1 and $B \in L^\infty(\mathbb{R}^{d+1},\mathbb{R}^{d+1})$, $\mathrm{div}\,B \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^{d+1})$, then there exists a unique $g \in L^\infty(\mathcal{H}^d\llcorner\partial\Omega)$ such that*

$$\big\langle \mathsf{Tr}^{\mathrm{in}}(B,\Omega)\cdot\mathbf{n},\psi\big\rangle = \int_{\partial\Omega'} g\psi\,\mathcal{H}^d, \qquad \forall\psi \in C^\infty_c(\mathbb{R}^{d+1}).$$

Sometimes we will refer to g as $\mathsf{Tr}^{\mathrm{in}}(B,\Omega)\cdot\mathbf{n} \in L^\infty(\mathcal{H}^d\llcorner\partial\Omega)$, with a slight abuse of notation. We collect here other important results on Anzellotti's weak traces:

Proposition 5.4. *Under the same assumptions of Proposition 5.3, it holds:*

- $\mathrm{div}\,B \ll \mathcal{H}^d$ *as measures in* $\mathbb{R}^{d+1}$*;*

- *if Σ is a C^1 hypersurface then*
$$\operatorname{div} B\llcorner_\Sigma = \big(\mathsf{Tr}^{\mathrm{out}}(B,\Omega)\cdot\mathbf{n} - \mathsf{Tr}^{\mathrm{in}}(B,\Omega)\cdot\mathbf{n}\big)\,\mathcal{H}^d\llcorner_\Sigma,$$
where Ω a set whose oriented boundary is Σ.

The orientation of Σ does not play a role in the second formula, because of (5.1). Finally, we recall the following proposition.

Proposition 5.5 (Fubini's Theorem for traces)**.** *Let B be as above and let $f \in C^1(\mathbb{R}^{d+1})$. Then*
$$\mathsf{Tr}^{\mathrm{in}}(B,\{f>t\})\cdot\mathbf{n} = B\cdot\mathbf{n}\,\mathcal{H}^d\llcorner_{\partial\{f>t\}} \quad \text{for } \mathcal{L}^1\text{-a.e. } t\in\mathbb{R},$$
where $\mathbf{n}$ denotes the exterior unit normal to $\{F>t\}$.

We remark that in the above results one can replace C^1 regularity with Lipschitz, see for example [ACM05, Remark 6.3].

5.1.3. *The non smooth setting.* We now drop the assumption that Ω has a regular boundary and we assume only that $\operatorname{div} B$ is a measure. We are going to prove (using Lagrangian representations) that the traces $\mathsf{Tr}^-(B,K)\cdot\mathbf{n}$ can be represented by a countable sum of Radon measures.

The case of one hitting time. To begin with, let us consider a simplified setting, i.e. assume that $|\mu|(\partial\Omega)=0$ and that there exists a well defined map

$$\begin{array}{rcccc} \mathtt{T} & : & \Gamma\supset\mathcal{D}(\mathtt{T}) & \to & I\times\partial\Omega \\ & & \gamma & \mapsto & \mathtt{T}(\gamma) := (t_\gamma,\gamma(t_\gamma)) \end{array} \tag{5.2}$$

such that $\gamma(t_\gamma)$ the unique point along the trajectory belonging to $\partial\Omega$ with (for the orientation)
$$(\mathrm{id},\gamma)([t^-_\gamma,t_\gamma))\in\Omega, \qquad (\mathrm{id},\gamma)((t_\gamma,t^+_\gamma])\in[0,T]\times\mathbb{R}^{d+1}\setminus\operatorname{clos}\Omega.$$
We assume moreover that a Lagrangian representation η is concentrated on $\mathcal{D}(\mathtt{T})$. In this case, we can prove the following

Proposition 5.6. *The distributions $\mathsf{Tr}^{\mathrm{in}}(B,\Omega)\cdot\mathbf{n}$ and $\mathsf{Tr}^{\mathrm{out}}(B,\Omega)\cdot\mathbf{n}$ are induced by a measure, i.e.*
$$\mathsf{Tr}^{\mathrm{in}}(B,\Omega)\cdot\mathbf{n} = \mathsf{Tr}^{\mathrm{out}}(B,\Omega)\cdot\mathbf{n} = \mathtt{T}_\sharp\eta,$$
where $\mathtt{T}$ is the map defined in (5.2).

Proof. By a direct computation, for any test function $\psi\in C^\infty_c((0,T)\times\mathbb{R}^d)$ it holds
$$\begin{aligned}\big\langle \mathsf{Tr}^{\mathrm{in}}(B,K)\cdot\mathbf{n},\psi\big\rangle &= \int_\Omega \psi\,\operatorname{div} B + \int_\Omega B\cdot\nabla\psi\,\mathcal{L}^{d+1} \\ &= \int_\Omega \psi\,\operatorname{div}(\rho(1,\mathbf{b})) + \int_\Omega \rho(1,\mathbf{b})\cdot\nabla\psi\,\mathcal{L}^{d+1} \\ &= \int \psi(t^-_\gamma,\gamma(t^-_\gamma))\,\eta(d\gamma) + \int\bigg[\int_{t^-_\gamma}^{t_\gamma}(1,\mathbf{b}(t,\gamma(t)))\cdot\nabla_{t,x}\psi(t,\gamma(t))\,dt\bigg]\eta(d\gamma) \\ &= \int \psi(t_\gamma,\gamma(t_\gamma))\,\eta(d\gamma),\end{aligned}$$
where we have used that η is a Lagrangian representation of $\rho(1,\mathbf{b})$. □

The general case: multiple hitting times. In the general case consider the open set
$$O := \big\{(t,\gamma):\gamma\in\Gamma,(t,\gamma(t))\in\Omega\big\}\subset\mathbb{R}\times\Gamma.$$
and decompose it as
$$O = \bigcup_{i,j\in\mathbb{N}}\big\{|t-t_i|<r_i\big\}\times\big\{\|\gamma-\gamma_j\|_{C^0}<r_j\big\} = \bigcup_{i,j,\in\mathbb{N}} B^1_{r_i}(t_i)\times B_{r_j}(\gamma_j).$$
For $\gamma\in B_{r_j}(\gamma_j)$ let $(t^{i,-}_\gamma,t^{i,+}_\gamma)$ be the connected component of $(\mathrm{id},\gamma)^{-1}(\Omega)$ such that
$$t_i\in\big(t^{i,-}_\gamma,t^{i,+}_\gamma\big).$$
It is elementary to show that $t^{i,+}_\gamma$ is l.s.c. and $t^{i,-}_\gamma$ is u.s.c. on $B_{r_j}(\gamma_j)$. We thus conclude that

Lemma 5.7. *There exists countably many Borel functions*

$$D_i \ni \gamma \mapsto t_\gamma^{i,-}, t_\gamma^{i,+}$$

such that

$$(\mathrm{id},\gamma)^{-1}(\Omega) = (t_\gamma^-, t_\gamma^{+,0}) \cup (t_\gamma^{-,0}, t_\gamma^+) \cup \bigcup_i (t_\gamma^{i,-}, t_\gamma^{i,+}),$$

where the first two intervals may be empty.

Proof. The only additional step is to relabel the intervals of $(\mathrm{id},\gamma)^{-1}(\Omega)$ which contains the initial time t_γ^- and the final time t_γ^+ as $t_\gamma^{+,0}$, $t_\gamma^{-,0}$, respectively. By the topology of $\varUpsilon$ this relabeling is still Borel. □

Trivially it holds for any test function $\psi \in C^\infty$

$$\begin{aligned}\int_{(\mathrm{id},\gamma)^{-1}(\Omega)} \frac{d}{dt}\psi(t,\gamma(t))\,dt = {}& \Big[\psi(t_\gamma^{+,0},\gamma(t_\gamma^{+,0})) - \psi(t_\gamma^-,\gamma(t_\gamma^-))\Big] + \Big[\psi(t_\gamma^+,\gamma(t_\gamma^+)) - \psi(t_\gamma^{-,0},\gamma(t_\gamma^{-,0}))\Big] \\ & + \sum_i \Big[\psi(t_\gamma^{i,+},\gamma(t_\gamma^{i,+})) - \psi(t_\gamma^{i,-},\gamma(t_\gamma^{i,-}))\Big],\end{aligned}$$

where the sum converges (as it is written) due to the estimate

$$\begin{aligned}\Big|\psi(t_\gamma^{i,+},\gamma(t_\gamma^{i,+})) - \psi(t_\gamma^{i,-},\gamma(t_\gamma^{i,-}))\Big| &\leq \|\nabla_{t,x}\psi\|_\infty\Big((t_\gamma^{i,+}-t_\gamma^{i,-}) + \big|\gamma(t_\gamma^{i,+}) - \gamma(t_\gamma^{i,-})\big|\Big) \\ &\leq \|\nabla_{t,x}\psi\|_\infty \int_{t_\gamma^{i,-}}^{t_\gamma^{i,+}} \big(1+|\dot\gamma(s)|\big)\,ds.\end{aligned} \tag{5.3}$$

It thus follows that

$$\begin{aligned}&\int_\Omega B\cdot\nabla_{t,x}\psi\,\mathcal{L}^{d+1} + \int_\Omega \psi\ \mathrm{div}\,B \\ &\quad = \int\bigg[\int_{(\mathrm{id},\gamma)^{-1}(\Omega)} \frac{d}{dt}\psi(t,\gamma(t))\,dt\bigg]\eta(d\gamma) \\ &\qquad + \int\Big[\psi(t_\gamma^-,\gamma(t_\gamma^-))\chi_\Omega(t_\gamma^-,\gamma(t_\gamma^-)) - \psi(t_\gamma^+,\gamma(t_\gamma^+))\chi_\Omega(t_\gamma^+,\gamma(t_\gamma^+))\Big]\eta(d\gamma) \\ &\quad = \int\bigg[\big(\psi(t_\gamma^{+,0},\gamma(t_\gamma^{+,0})) - \psi(t_\gamma^{-,0},\gamma(t_\gamma^{-,0})\big) + \sum_i\Big[\psi(t_\gamma^{i,+},\gamma(t_\gamma^{i,+})) - \psi(t_\gamma^{i,-},\gamma(t_\gamma^{i,-}))\Big]\bigg]\eta(d\gamma).\end{aligned}$$

Thanks to (5.3), we can partition the last sum as

$$\begin{aligned}\int\sum_i\big[\psi(t_\gamma^{i,+},\gamma(t_\gamma^{i,+})) - \psi(t_\gamma^{i,-},\gamma(t_\gamma^{i,-}))\big]\eta(d\gamma) &= \sum_i\int\big[\psi(t_\gamma^{i,+},\gamma(t_\gamma^{i,+})) - \psi(t_\gamma^{i,-},\gamma(t_\gamma^{i,-}))\big]\eta(d\gamma) \\ &= \sum_i\big\langle(\mathtt{T}_\Omega^{i,+})_\sharp\eta - (\mathtt{T}_\Omega^{i,-})_\sharp\eta,\psi\big\rangle,\end{aligned}$$

where

$$\mathtt{T}_\Omega^{i,\pm} : \gamma \mapsto (t_\gamma^{i,\pm},\gamma(t_\gamma^{i,\pm})) \in \partial\Omega. \tag{5.4}$$

We thus have obtained the following lemma.

Lemma 5.8. *The distributional trace of $B = \rho(1,\mathbf{b})$ can be represented as the countable sum of measures supported on $\partial\Omega$, namely*

$$\mathsf{Tr}^{\mathrm{in}}(\rho(1,\mathbf{b}),\Omega)\cdot\mathbf{n} = \sum_{i=0}^{\mathbb{N}}(\mathtt{T}_\Omega^{i,+})_\sharp\eta - (\mathtt{T}_\Omega^{i,-})_\sharp\eta \tag{5.5}$$

and the series convergences in the sense of distributions.

Define now the *restriction operators* $\mathtt{R}_\Omega^i$, $\mathtt{R}_\Omega$ as

$$\mathtt{R}_\Omega^i\gamma := \gamma\llcorner_{(t_i^-(\gamma),t_i^+(\gamma))}, \qquad \mathtt{R}_\Omega\gamma = \big\{\mathtt{R}_\Omega^i\gamma\big\}_i, \tag{5.6}$$

and the measures η_Ω^i as

$$\eta_\Omega^i := (\mathtt{R}_\Omega^i)_\sharp\eta. \tag{5.7}$$

See Figure 6.

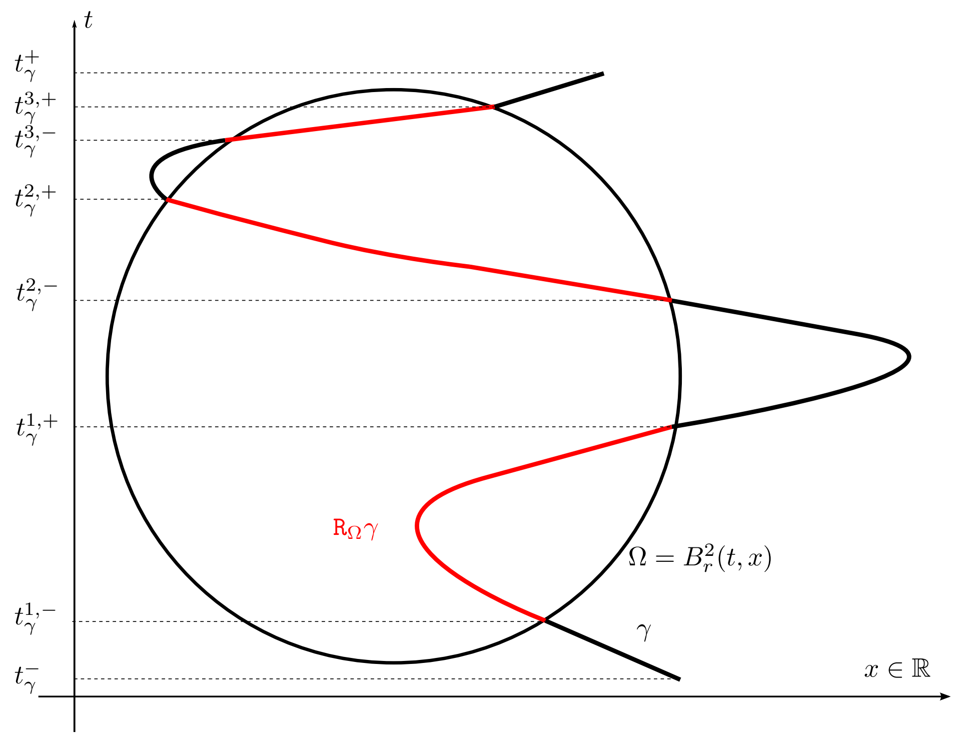


Figure 6. Restriction operator $\mathtt{R}_\Omega$ in the case Ω is a ball $B^2_r(t,x)$. The curve γ (depicted in black) is cut into the three red pieces which make up $\mathtt{R}_\Omega\gamma$.

It is clear that if

$$\rho^i_\Omega(1,\mathbf{b})\,\mathcal{L}^{d+1} := \int (\mathrm{id},\gamma)_\sharp\big((1,\dot\gamma)\,\mathcal{L}^1\big)\,\eta^i_\Omega(d\gamma), \tag{5.8}$$

then in Ω

$$\rho(1,\mathbf{b}) = \sum_i \rho^i_\Omega(1,\mathbf{b})$$

and

$$\mathsf{Tr}^{\mathrm{in}}(\rho^i_\Omega(1,\mathbf{b}),\Omega) = (\mathtt{T}^{i,+}_\Omega)_\sharp\eta^i_\Omega - (\mathtt{T}^{i,-}_\Omega)_\sharp\eta^i_\Omega = \mathrm{div}_{t,x}\big(\rho^i_\Omega(1,\mathbf{b})\big).$$

We remark that even if for η^i_Ω the series in (5.5) reduces to a finite sum of measures, the measure η^i_Ω is not in general a Lagrangian representation of $\rho^i_\Omega(1,\mathbf{b})$, unless

$$(\mathtt{T}^{i,+}_\Omega)_\sharp\eta^i_\Omega \perp (\mathtt{T}^{i,-}_\Omega)_\sharp\eta^i_\Omega.$$

Example 5.9. One can construct a vector field $\mathbf{b} \in L^\infty(\mathbb{R}^3)$ supported in $[-1,0]\times[0,1]^2)$ with the following properties:

(1) it is divergence-free, smooth outside $\{x_1 = 1\}$ and of the form $(1,\tilde b(x_1,x^\perp))$, $(x,x^\perp) \in \mathbb{R}\times\mathbb{R}^2$;
(2) the flow $\tilde X$ generated by the ODE

$$\frac{d\tilde X}{dx_1} = \mathbf{b}(x_1,\tilde X), \qquad \tilde X(-1,x^\perp) = x^\perp,$$

has the property that it can be extended by continuity to $x_1 = 0$ and it holds

$$(\tilde X(0))_\sharp\big(\mathcal{L}^2\llcorner_{(0,1/2)\times(0,1)}\big) = (\tilde X(0))_\sharp\big(\mathcal{L}^2\llcorner_{(1/2,1)\times(0,1)}\big) = \frac{1}{2}\mathcal{L}^2\llcorner_{(0,1)^2}.$$

The above assumptions yields that there exists a solution to

$$\mathrm{div}_x\big(\tilde\rho(1,\tilde{\mathbf{b}})\big) = 0$$

which is w^*-continuous in L^∞ w.r.t. x_1 and such that

$$\tilde\rho(-1,x^\perp) = \mathbb{1}_{(1/2,1)\times(0,1)}(x^\perp) - \mathbb{1}_{(0,1/2)\times(0,1)}(x^\perp), \quad \tilde\rho(x_1<0)\in\{-1,1\}, \quad \tilde\rho(x_1>0) = 0.$$

An example of a construction can be found in [ACM05, Example 3.8], see also [Dep03].

Define the vector field

$$\mathbf{b}(x_1, x^\perp) = (\tilde{\rho}\tilde{\mathbf{b}})(x_1, x^\perp),$$

so that it is divergence free, and its trace on $\{x_1 = 0\}$ is 0. In particular $\rho^- := \mathbb{1}_{\{x_1<0\}}$ is a solution to $\mathrm{div}_{t,x}(\rho(1,\mathbf{b})) = 0$.

Let η^- be a Lagrangian representation for $\rho^-(1,\mathbf{b})$: due to the uniqueness of $\bar{X}$, the set of curves on which η is concentrated is the set of curves such that, if t_γ is the time where $\gamma(t_\gamma) \in \{x_1 = 0\}$, then

$$\gamma(t) = \begin{cases} \bar{X}\big(1 + (t - t_\gamma), x_\gamma^{\perp,-}\big) & t < t_\gamma, \\ \bar{X}\big(1 - (t - t_\gamma), -x_\gamma^{\perp,+}\big) & t > t_\gamma, \end{cases}$$

with $x_\gamma^{\perp,-} \in (1/2, 1) \times (0, 1)$, $x_\gamma^{\perp,+} \in (0, 1/2) \times (0, 1)$.

If now we extend the vector field $\mathbf{b}$ to the region $x_1 > 0$ by symmetry

$$\mathbf{b}(x_1, x^\perp) = -\mathbf{b}(-x_1, x^\perp),$$

then a Lagrangian representation η is obtained by adding η^- with

$$\eta^+ = S_\sharp \eta^-,$$

where $S(\gamma)$ is the symmetric curve w.r.t. $\{x_1 = 0\}$,

$$S(\gamma)(t) = (-\gamma_1, \gamma^\perp)(t).$$

Now we can construct a new Lagrangian representation η' for the extended $(1,\mathbf{b})$ by piecing together the curves γ and $S(\gamma)$ in order to let both cross the surface: more precisely, defining the maps

$$\left.\begin{matrix} \bar{X}\big(1 + (t - t_\gamma), x_\gamma^{\perp,-}\big) & t < t_\gamma \\ \bar{X}\big(1 - (t - t_\gamma), -x_\gamma^{\perp,+}\big) & t > t_\gamma \end{matrix}\right\} = \gamma \mapsto G_1(\gamma) = \left\{ \begin{matrix} \bar{X}\big(1 + (t - t_\gamma), x_\gamma^{\perp,-}\big) & t < t_\gamma \\ (-\bar{X}, \bar{X}^\perp)\big(1 - (t - t_\gamma), -x_\gamma^{\perp,+}\big) & t > t_\gamma \end{matrix}\right.$$

$$\left.\begin{matrix} \bar{X}\big(1 + (t - t_\gamma), x_\gamma^{\perp,-}\big) & t < t_\gamma \\ \bar{X}\big(1 - (t - t_\gamma), -x_\gamma^{\perp,+}\big) & t > t_\gamma \end{matrix}\right\} = \gamma \mapsto G_2(\gamma) = \left\{ \begin{matrix} (-\bar{X}, \bar{X}^\perp)\big(1 + (t - t_\gamma), x_\gamma^{\perp,-}\big) & t < t_\gamma \\ \bar{X}\big(1 - (t - t_\gamma), -x_\gamma^{\perp,+}\big) & t > t_\gamma \end{matrix}\right.$$

the Lagrangian representation is given by

$$\eta' = (G_1)_\sharp \eta^- + (G_2)_\sharp \eta^-.$$

A simple computation yields for η' it holds

$$(\mathtt{T}^{0,+}_{\{x_1<0\}})_\sharp \eta' = (\mathtt{T}^{0,-}_{\{x_1<0\}})_\sharp \eta' = \|\eta'\|,$$

while being $\mathsf{Tr}^{\mathrm{in}}(\mathbf{b}, \{x_1 < 0\}) \cdot \mathbf{n} = 0$ both terms should be 0.

A small variation of the above example (i.e. letting the curves cross the surface several times) shows that the sum (5.5) is diverging in the general case.

5.2. **Bounded variation vector fields.** Before considering a general vector fields $\rho(1,\mathbf{b}) \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1})$, we improve the regularity of ρ, $\mathbf{b}$ so that the restriction operator $\mathtt{R}_\Omega$ preserves the property of being a Lagrangian representation if Ω has a Lipschitz boundary.

Let $\Omega \subset \mathbb{R}^{d+1}$ be an open set with a Lipschitz boundary $\partial\Omega$ and assume that $\mathbf{b} \in (L^1_{\mathrm{loc}})_t((\mathrm{BD}_{\mathrm{loc}})_x)$. Let $\rho \in L^\infty$ be a positive solution to $\mathrm{div}_{t,x}(\rho(1,\mathbf{b})) = \mu$ and let η be an associated Lagrangian representation.

We recall that BD-functions $\mathbf{b}$ have a full inner trace on open sets $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, i.e. there exists a vector valued measure which is a.c. w.r.t. $\mathcal{H}^{d-1} \llcorner_{\partial\Omega}$, which we we will denote it by

$$\mathsf{Tr}^{\mathrm{in}}(\mathbf{b}, \Omega)\, \mathcal{H}^d,$$

with a slight abuse of notation, and such that

$$\lim_{r \searrow 0} \fint_{B^d_r(x)\cap\Omega} \big|\mathbf{b}(\tau, y) - \mathsf{Tr}^{\mathrm{in}}(\mathbf{b}, \Omega)\big|\, dy = 0.$$

for $\mathcal{H}^d$-a.e. $x \in \partial\Omega$ (see [Bab15]). The following result holds (see [ACM05, Theorem 4.2] and subsequent remarks).

Theorem 5.10 (Change of variables for traces)**.** *Let $\Omega \subset \mathbb{R}^{d+1}$ be an open domain with a Lipschitz boundary and let $\beta \in \mathrm{Lip}(\mathbb{R})$. Then the trace of $\rho(1,\mathbf{b})$ is a.c. w.r.t. $\mathcal{H}^d \llcorner_{\partial\Omega}$ and*

$$\mathsf{Tr}^{\mathrm{in}}\big(\beta(\rho)\mathbf{b},\Omega\big)\cdot\mathbf{n} = \beta\bigg(\frac{\mathsf{Tr}^{\mathrm{in}}(\rho\mathbf{b},\Omega)\cdot\mathbf{n}}{\mathsf{Tr}^{\mathrm{in}}(\mathbf{b},\Omega)\cdot\mathbf{n}}\bigg)\,\mathsf{Tr}^{\mathrm{in}}\big(\mathbf{b},\Omega\big)\cdot\mathbf{n} \qquad \mathcal{H}^d\text{-a.e. on } \partial\Omega,$$

where the ratio is arbitrarily defined at points where the trace $\mathsf{Tr}^{\mathrm{in}}(\mathbf{b},\Omega)\cdot\mathbf{n}$ vanishes.

Again we have written $\mathsf{Tr}^{\mathrm{in}}\big(\beta(\rho)\mathbf{b},\Omega\big)\cdot\mathbf{n}$ as the density of the inner trace w.r.t. $\mathcal{H}^d$, no confusion should occur.

Consider now $\Omega \subset \mathbb{R}^{d+1}$ with Lipschitz boundary, and let $\mathbf{n}$ be the outer normal defined $\mathcal{H}^d\llcorner_{\Omega}$-a.e.. Then the following slight extension of the above theorem holds.

Proposition 5.11. *The trace of the vector fields $\rho(1,\mathbf{b})$ is a.c. w.r.t. $\mathcal{H}^d\llcorner_{\partial\Omega}$ and*

$$\mathsf{Tr}^{\mathrm{in}}\big(\beta(\rho)(1,\mathbf{b}),\Omega\big)\cdot\mathbf{n} = \beta\bigg(\frac{\mathsf{Tr}^{\mathrm{in}}(\rho(1,\mathbf{b}),\Omega)\cdot\mathbf{n}}{\mathsf{Tr}^{\mathrm{in}}((1,\mathbf{b}),\Omega)\cdot\mathbf{n}}\bigg)\,\mathsf{Tr}^{\mathrm{in}}\big((1,\mathbf{b}),\Omega\big) \qquad \mathcal{H}^d\text{-a.e. on } \partial\Omega.$$

Proof. The proof proceeds as for the previous theorem, with some easy generalizations: we will only sketch it. It is not restrictive to localize the problem in a large ball $B^{d+1}_R(0,0)$.

The fact that

$$\mathsf{Tr}^{\mathrm{in}}\big((1,\mathbf{b}),\Omega\big) = \int\big[-\partial_t \mathbb{1}_{\Omega_x}\mathbf{e}_1\big]\,\mathcal{L}^d(dx) + \int\big[\mathsf{Tr}^{\mathrm{in}}(\mathbf{b}_t,\Omega_t)\,\mathcal{H}^{d-1}\llcorner_{\partial\Omega_t}\big]\,dt \ll \mathcal{H}^d\llcorner_{\partial\Omega}$$

is a consequence of the linearity of trace, Fubini's Theorem and Coarea Formula, while using the strong convergence of traces for BD vector fields one can show that

$$\mathsf{Tr}^{\mathrm{in}}\big(\rho(1,\mathbf{b}),\Omega\big)\cdot\mathbf{n} = \theta\,\mathsf{Tr}^{\mathrm{in}}\big((1,\mathbf{b}),\Omega\big)\,\mathcal{H}^d,$$

with $\theta \in L^\infty(\mathcal{H}^d)$ (actually $\|\theta\|_\infty \le \|\rho\|_\infty$). Recall that $\Omega_{\bar t} = \Omega \cap \{t = \bar t\}$, $\Omega_{\bar x} = \Omega \cap \{x = \bar x\}$.

Step 1. First note that by the classical computation for renormalized solutions it holds

$$\partial_t\beta(\rho) + \operatorname{div}\big(\beta(\rho)\mathbf{b}\big) \ll |\mu| + \int |\mathrm{E}\mathbf{b}|\,dt,$$

with $\mathrm{E}\mathbf{b}$ the symmetric part of $D\mathbf{b}$. Indeed, disintegrate

$$\mu = \int \mu_t\,m(dt), \qquad m = (\mathbf{p}_t)_\sharp|\mu|,$$

and decompose m in its continuous m^{cont} and atomic part $m^{\mathrm{atomic}} = \sum_i m_i\delta_{t_i}$. If $\varphi^\varepsilon \in C^\infty(\mathbb{R}^d)$ is a smooth compactly supported convolution kernel in $\mathbb{R}^d$, then

$$\partial\rho^\varepsilon + \operatorname{div}(\rho b)^\varepsilon = \int \mu^\varepsilon_t\,m(dt),$$

where with the apex ε we mean the convolution with φ^ε. The chain rule for one-dimensional BV functions thus yields for $\beta \in C^1$

$$\begin{aligned}
\partial_t\beta(\rho^\varepsilon) + \operatorname{div}\big(\beta(\rho^\varepsilon)\mathbf{b}\big) &= \int \beta'(\rho^\varepsilon(t))(\mu^\varepsilon_t)\,m^{\mathrm{cont}}(dt) + \sum_i\big[\beta\big(\rho^\varepsilon(t_i-) + c_i\mu^\varepsilon_{t_i}\big) - \beta(\rho^\varepsilon(t_i-))\big] \\
&\quad + \operatorname{div}\big(\beta(\rho^\varepsilon)\mathbf{b}\big) - \beta'(\rho^\varepsilon)\big(\operatorname{div}(\rho\mathbf{b})\big)^\varepsilon \\
&= \tilde\mu^\varepsilon + \beta(\rho^\varepsilon)\operatorname{div}\mathbf{b} \\
&\quad + \beta'(\rho^\varepsilon)\int \rho(t,x-\varepsilon y)\frac{\mathbf{b}(t,x) - \mathbf{b}(t,x-\varepsilon y)}{\varepsilon}\cdot\nabla\varphi(y)\,\mathcal{L}^{d+1}(dtdy),
\end{aligned}$$

with

$$\tilde\mu^\varepsilon = \int \beta'(\rho^\varepsilon(t))(\mu^\varepsilon_t)\,m^{\mathrm{cont}}(dt) + \sum_i\big(\beta\big[\rho^\varepsilon(t_i-) + c_i\mu^\varepsilon_{t_i}\big) - \beta(\rho^\varepsilon(t_i-))\big].$$

Clearly $|\tilde\mu^\varepsilon| \le \|\beta'\circ\rho\|_\infty|\mu|$.

The last term of the r.h.s. is bounded by (assuming $\operatorname{supp}\phi^\varepsilon \subset B^d_1(0)$)

$$\mathcal{O}(1)\int_{B^d_1(0)}\bigg|\frac{\mathbf{b}_t(x) - \mathbf{b}_t(x-\varepsilon y)}{\varepsilon}\cdot y\bigg|\,\mathcal{L}^d(dy),$$

which converges weakly to

$$\mathcal{O}(1) \int |\langle \mathrm{E}\mathbf{b}_t z, z\rangle| \, dt.$$

Since the constant depends only on $\|\beta'\|_\infty$, the same estimate holds for β Lipschitz.

Step 2. The formula in the statement holds for ρ, $\mathbf{b}$ in $W^{1,1} \cap L^\infty(\mathbb{R}^{d+1})$ by the strong continuity of traces, so that it is enough to show that for all $\delta > 0$ it is possible to extend $\rho(1, \mathbf{b})$ in $\mathbb{R}^{d+1} \setminus \Omega$ with a $W^{1,1} \cap L^\infty$-vector field $(1, \mathbf{b}')$ and $W^{1,1} \cap L^\infty$-function ρ' such that

$$\Big\| \mathsf{Tr}^{\mathrm{in}} \big(\rho'(1, \mathbf{b}'), \mathbb{R}^{d+1} \setminus \operatorname{clos} \Omega\big) \cdot \mathbf{n} - \mathsf{Tr}^{\mathrm{in}} \big(\rho(1, \mathbf{b}), \Omega\big) \cdot \mathbf{n} \Big\|_{L^1(\mathcal{H}^d \llcorner_{\partial\Omega})} < \delta, \tag{5.9a}$$

$$\Big\| \mathsf{Tr}^{\mathrm{in}} \big((1, \mathbf{b}'), \mathbb{R}^{d+1} \setminus \operatorname{clos} \Omega\big) - \mathsf{Tr}^{\mathrm{in}} \big((1, \mathbf{b}), \Omega\big) \Big\|_{L^1(\mathcal{H}^d \llcorner_{\partial\Omega})} < \delta, \tag{5.9b}$$

where the last formula means that the traces coincide as vectors, or equivalently in the sense of boundary traces of BD functions.

Indeed, setting

$$\rho''(1, \mathbf{b}'') := \rho(1, \mathbf{b}) \mathbb{1}_\Omega + \rho'(1, \mathbf{b}') \mathbb{1}_{\mathbb{R}^d \setminus \operatorname{clos} \Omega},$$

by (5.9a) it follows that

$$\big| \operatorname{div} \big(\rho''(1, \mathbf{b}'')\big) \big| (\partial\Omega) < \delta,$$

and (5.9b) yields that $|\mathrm{E}\mathbf{b}''|(\partial\Omega) < \delta$. Thus by Step 1 the outer and inner trace of $\beta(\rho)(1, \mathbf{b}'')$ differs from the outer trace by $\mathcal{O}(1)\delta$ in norm, where the $W^{1,1}$-computation can be performed. Letting $\delta \to 0$ and using a pointwise convergent subsequence $\rho', \mathbf{b}'$ in (5.9), one obtains the formula in the statement.

Step 3. Being Ω Lipschitz, it follows that for $\mathcal{L}^1$-a.e. $\bar t \in \mathbb{R}$ the set $\Omega_{\bar t}$ is of Lipschitz regularity, and thus by the surjectivity of traces of $W^{1,1}$ into L^1 let $\mathbf{b}'_{\bar t}$ be an extension on $\mathbb{R}^d \setminus \Omega_{\bar t}$ such that

$$\mathsf{Tr}^{\mathrm{in}} \big(\mathbf{b}', \partial(\mathbb{R}^d \setminus \operatorname{clos} \Omega_t)\big) = \mathsf{Tr}^{\mathrm{in}} \big(\mathbf{b}, \partial\Omega_t\big)\llcorner_{\mathsf{Tr}^{\mathrm{in}}(\mathbf{b}, \partial\Omega_t) < 2^n} \quad \mathcal{L}^1\text{-a.e. } t.$$

By inspection of the proof of Gagliardo's theorem [Gag57, Theorem 1.II], one can check that we can also require that $\mathbf{b}' \in L^1_{t,\mathrm{loc}}(\mathrm{BV}_x) \cap L^\infty$, because

$$\mathsf{Tr}^{\mathrm{in}}((1, \mathbf{b}), \Omega)\llcorner_{\mathsf{Tr}^{\mathrm{in}}(\mathbf{b}, \partial\Omega_t) < 2^n} \in L^1_{\mathrm{loc}}(\mathcal{H}^d \llcorner_{\partial\Omega}) \cap L^\infty.$$

It is fairly easy to see from the definition of trace that (5.9b) holds.

Step 4. Being the function

$$\theta = \frac{\mathsf{Tr}(\rho(1, \mathbf{b}), \partial\Omega)}{\mathsf{Tr}((1, \mathbf{b}), \partial\Omega)}$$

in $L^\infty(\mathcal{H}^d \llcorner_{\partial\Omega})$, again by Gagliardo's theorem there is $w' \in W^{1,1} \cap L^\infty(\mathbb{R}^{d+1} \setminus \Omega)$ such that (5.9a) holds. Being $\mathbf{b}'$ and ρ' bounded functions, we are in the setting of Step 2 above. □

Using the above theorem we show that the restriction of a Lagrangian representation in the sense of (5.6) is a Lagrangian representation of the vector field $\rho(1, \mathbf{b}) \mathcal{L}^d \llcorner_\Omega$ for vector fields in $\mathbf{b} \in L^1_t(\mathrm{BD}_x)$ and functions $\rho \in L^\infty(\mathbb{R}^{d+1})$.

Proposition 5.12. *The measure*

$$(\mathtt{R}_\Omega)_\sharp \eta := \sum_i \eta^i_\Omega = \sum_i (\mathtt{R}^i_\Omega)_\sharp \eta$$

is a Lagrangian representation of $\rho(1, \mathbf{b}) \mathcal{L}^{d+1} \llcorner_\Omega$.

Proof. Let ρ^i_Ω be defined as in (5.8); in particular, the distribution $\mathsf{Tr}^{\mathrm{in}}(\rho^i_\Omega(1, \mathbf{b}), \Omega) \cdot \mathbf{n}$ is now representable as sum of two Radon measures $(\mathtt{T}^{i,\pm}_\Omega)_\sharp \eta^i_\Omega$, being $\mathtt{T}^{i,\pm}_\Omega$ defined as in (5.4).

By applying now Theorem 5.10 with $\beta(\cdot) = |\cdot|$, we deduce the following:

$$\mathsf{Tr}^{\mathrm{in}} \big(\rho^i_\Omega(1, \mathbf{b}), \partial\Omega\big) \cdot \mathbf{n} = \mathsf{Tr}^{\mathrm{in}} \big(|\rho^i_\Omega|(1, \mathbf{b}), \partial\Omega\big) \cdot \mathbf{n} = \left| \frac{\mathsf{Tr}^{\mathrm{in}} \big(\rho^i_\Omega(1, \mathbf{b}), \partial\Omega\big) \cdot \mathbf{n}}{\mathsf{Tr}^{\mathrm{in}} \big((1, \mathbf{b}), \partial\Omega\big) \cdot \mathbf{n}} \right| \mathsf{Tr}^{\mathrm{in}} \big((1, \mathbf{b}), \partial\Omega\big) \cdot \mathbf{n}$$

because $\rho^i_\Omega \geq 0$. It thus follows that $\mathsf{Tr}^{\mathrm{in}}(\rho^i_\Omega(1, \mathbf{b}), \partial\Omega) \cdot \mathbf{n}$ has the same sign of $\mathsf{Tr}^{\mathrm{in}}((1, \mathbf{b}), \partial\Omega) \cdot \mathbf{n}$, which means that $(\mathtt{T}^{i,\pm}_\Omega)_\sharp \eta^i_\Omega$ are orthogonal.

Hence there exists two disjoint Borel sets $A^\pm$ such that for all $i \in \mathbb{N}$

$$\mathsf{Tr}^{\mathrm{in},\pm} \big(\rho^i_\Omega(1, \mathbf{b}), \partial\Omega\big) \cdot \mathbf{n} = \big((\mathtt{T}^{i,\pm}_\Omega)_\sharp \eta^i_\Omega\big)\llcorner_{A^\pm},$$

where $A^\pm$ are determined by

$$\mathsf{Tr}^{\mathrm{in},\pm}\big((1,\mathbf{b}),\partial\Omega\big)\cdot\mathbf{n} = \mathsf{Tr}^{\mathrm{in}}\big((1,\mathbf{b}),\partial\Omega\big)\cdot\mathbf{n}\llcorner_{A^\pm},$$

up to $\mathsf{Tr}^{\mathrm{in}}((1,\mathbf{b}),\partial\Omega)\cdot\mathbf{n}$-negligible sets. Here the apex $\pm$ means the positive/negative part of the trace.

Furthermore, repeating the argument for a finite sum of η^i_Ω it follows

$$\sum_i^N \rho^i_\Omega \le \rho,$$

and

$$\begin{aligned}
\sum_i^N \mathsf{Tr}^{\mathrm{in},\pm}\big(\rho^i_\Omega(1,\mathbf{b}),\partial\Omega\big)\cdot\mathbf{n} &= \sum_i^N \mathsf{Tr}^{\mathrm{in}}\big(\rho^i_\Omega(1,\mathbf{b}),\partial\Omega\big)\cdot\mathbf{n}\llcorner_{A^\pm} \\
&= \bigg(\sum_i^N \mathsf{Tr}^{\mathrm{in}}\big(\rho^i_\Omega(1,\mathbf{b}),\Omega\big)\cdot\mathbf{n}\bigg)\llcorner_{A^\pm} \\
&= \bigg(\mathsf{Tr}^{\mathrm{in}}\bigg(\sum_i^N \rho^i_\Omega(1,\mathbf{b}),\Omega\bigg)\cdot\mathbf{n}\bigg)\llcorner_{A^\pm} \\
&= \mathsf{Tr}^{\mathrm{in},\pm}\bigg(\sum_i^N \rho^i_\Omega(1,\mathbf{b}),\Omega\bigg)\cdot\mathbf{n} \\
&\le \mathsf{Tr}^{\mathrm{in},\pm}\big(\rho(1,\mathbf{b}),\Omega\big)\cdot\mathbf{n},
\end{aligned}$$

where we have used the monotonicity of the trace (consequence of Proposition 5.11). It follows that

$$\sum_i \mathsf{Tr}^{\mathrm{in},\pm}\big(\rho^i_\Omega(1,\mathbf{b}),\Omega\big)\cdot\mathbf{n} = \mathsf{Tr}^{\mathrm{in},\pm}\big(\rho(1,\mathbf{b}),\Omega\big)\cdot\mathbf{n} < +\infty,$$

where the equality follows from the weak convergence of the sum to the trace. □

Another consequence of Proposition 5.11 is the following (see also [ACM05, Theorem 6.2]. We consider here the definition of inner proper, Definition 4.5, extended to Lipschitz sets as suggested in Remark 4.6.

Corollary 5.13. *A Lipschitz open set Ω is inner proper for the vector fields $\rho(1,\mathbf{b})$, with $\rho \in L^\infty(\mathbb{R}^{d+1})$ and $\mathbf{b} \in L^1_t(\mathrm{BD}_x)$.*

Proof. Indeed, the chain rule and the strong convergence of traces for BD vector fields yields that Condition (3) of Definition 4.5 holds. □

6. Restriction operator $\mathtt{R}$ and proper sets

We now show that for generic vector fields $\rho(1,\mathbf{b}) \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1})$, if Ω is a $\rho(1,\mathbf{b})$-proper set, then the reduction operator $\mathtt{R}_\Omega$ introduced in Proposition 5.12, namely

$$(\mathtt{R}_\Omega)_\sharp\eta := \sum_i (\mathtt{R}^i_\Omega)_\sharp\eta, \tag{6.1}$$

generates a Lagrangian representation of $\rho(1,\mathbf{b})\mathcal{L}^{d+1}\llcorner_\Omega$. The idea of the proof is to show that there are two disjoint sets where η-a.e. curve γ is only entering or exiting. We conclude this section with some useful properties of the operator $\mathtt{R}_\Omega$ for proper sets.

We begin with the following elementary lemma.

Lemma 6.1. *For every Lipschitz function $0 \le \psi \le 1$ it holds*

$$\eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \{\psi = 1\} \neq \emptyset,\ \mathrm{Graph}\,\gamma \cap \{\psi = 0\} \neq \emptyset\Big\}\Big) \le \int \rho\big|(1,\mathbf{b})\cdot\nabla\psi\big|\mathcal{L}^{d+1}.$$

Proof. Setting

$$A := \Big\{\gamma : \mathrm{Graph}\,\gamma \cap \{\psi = 1\} \neq \emptyset,\ \mathrm{Graph}\,\gamma \cap \{\psi = 0\} \neq \emptyset\Big\},$$

one has for $\gamma \in A$

$$\int_{t_\gamma^-}^{t_\gamma^+} \big|(1,\mathbf{b})\cdot\nabla\psi\big|\,dt = \mathrm{Tot.Var.}\psi(\gamma) \geq 1,$$

so that

$$\eta(A) \leq \int_A \mathrm{Tot.Var.}\big(\psi\circ\gamma\big)\,\eta(d\gamma) \leq \int \rho\big|(1,\mathbf{b})\cdot\nabla\psi\big|\,\mathcal{L}^{d+1}$$

which concludes the proof. □

Applying Lemma 6.1 to a proper set Ω with the functions $\phi^{\delta,\pm}$ and passing to the limit as $\delta \to 0$ we obtain the following

Proposition 6.2. *It holds*

$$\eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \mathrm{clos}\,\Omega \neq \emptyset, \mathrm{Graph}\,\gamma \cap \mathbb{R}^{d+1}\setminus \mathrm{clos}\,\Omega \neq \emptyset\Big\}\Big) \leq \int_{\partial\Omega} \rho\big|(1,\mathbf{b})\cdot\mathbf{n}\big|\,\mathcal{H}^d, \tag{6.2}$$

and

$$\eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \Omega \neq \emptyset, \mathrm{Graph}\,\gamma \cap \mathbb{R}^{d+1}\setminus \Omega \neq \emptyset\Big\}\Big) \leq \int_{\partial\Omega} \rho\big|(1,\mathbf{b})\cdot\mathbf{n}\big|\,\mathcal{H}^d. \tag{6.3}$$

In particular, for every proper set we deduce that

$$\begin{aligned}
&\eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \partial\Omega \neq \emptyset, \mathrm{Graph}\,\gamma \not\subseteq \partial\Omega\Big\}\Big) \\
&\quad\leq \eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \Omega \neq \emptyset, \mathrm{Graph}\,\gamma \cap \mathbb{R}^{d+1}\setminus\Omega \neq \emptyset\Big\}\Big) \\
&\qquad+ \eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \mathrm{clos}\,\Omega \neq \emptyset, \mathrm{Graph}\,\gamma \cap \mathbb{R}^{d+1}\setminus\mathrm{clos}\,\Omega \neq \emptyset\Big\}\Big) \\
&\quad\leq 2\int_{\partial\Omega} \rho\big|(1,\mathbf{b})\cdot\mathbf{n}\big|\mathcal{H}^d.
\end{aligned} \tag{6.4}$$

At the end of this section Corollary 6.9 gives that the constant 2 can be replaced with 1.

Let Ω be a proper set and let Ω^ε its perturbation constructed in Theorem 4.16: moreover, if $K_{\bar r}^{\varepsilon,\varepsilon'} \subset \partial\Omega$ is the compact set constructed in Lemma 4.11, w.l.o.g. we can assume that $\rho(1,\mathbf{b})\llcorner_{K_{\bar r}^{\varepsilon,\varepsilon'}}$ is continuous. Recall the decomposition

$$\partial\big(\Omega^\varepsilon\setminus\Omega\big) = S_1 \cup S_2 \cup S_3^+ \cup S_3^- \cup S_4$$

given in (4.13), where S_1, S_2 are subset of finitely many hyperplanes $\{t = \mathrm{const}\}$, and S_4 is a subset of the lateral faces of the cylinders given by Proposition 4.15.

Applying (6.4) to the lateral boundary of a cylinder

$$\partial^l\mathrm{Cyl}_{t_i,x_i}^{r_i,2\alpha^2} = \Big\{(s,y) : |s-t_i| \leq 2\alpha^2 r_i, |y - x_i - \mathbf{b}(t_i,x_i)(s-t_i)| = r_i\Big\},$$

and considering the trajectories restricted to

$$J_\gamma^i := [t_\gamma^-, t_\gamma^+] \cap \big[t_i - 2\alpha^2 r_i, t_i + 2\alpha^2 r_i\big],$$

we obtain

$$\begin{aligned}
&\eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \partial^l\mathrm{Cyl}_{t_i,x_i}^{r_i,2\alpha^2} \neq \emptyset, \mathrm{Graph}\,\gamma \cap (J_\gamma^i \times \mathbb{R}^d) \not\subseteq \partial^l\mathrm{Cyl}_{t_i,x_i}^{r_i,2\alpha^2}\Big\}\Big) \\
&\leq \eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma\llcorner_{J_\gamma^i} \cap \mathrm{Cyl}_{t_i,x_i}^{r_i,2\alpha^2} \neq \emptyset, \mathrm{Graph}\,\gamma\llcorner_{J_\gamma^i} \cap \big(\mathbb{R}^{d+1}\setminus\mathrm{Cyl}_{t_i,x_i}^{r_i,2\alpha^2}\big) \neq \emptyset\Big\}\Big) \\
&\quad+ \eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma\llcorner_{J_\gamma^i} \cap \mathrm{clos}\,\mathrm{Cyl}_{t_i,x_i}^{r_i,2\alpha^2} \neq \emptyset, \mathrm{Graph}\,\gamma\llcorner_{J_\gamma^i} \cap \big(\mathbb{R}^{d+1}\setminus\mathrm{clos}\,\mathrm{Cyl}_{t_i,x_i}^{r_i,2\alpha^2}\big) \neq \emptyset\ \Big\}\Big) \\
&\leq 2\int_{\partial\mathrm{Cyl}_{t_i,x_i}^{r_i,2\alpha^2}} \rho\big|(1,\mathbf{b})\cdot\mathbf{n}\big|\mathcal{H}^d.
\end{aligned} \tag{6.5}$$

Then we can prove the following.

Lemma 6.3. *It holds*

$$\eta\big(\big\{\gamma : \mathrm{Graph}\,\gamma \cap S_4 \neq \emptyset\big\}\big) \leq 2(1+2\alpha)C_d\varepsilon' L^d\mathcal{H}^d(\partial\Omega).$$

Proof. We observe that

$$\big\{\gamma : \mathrm{Graph}\,\gamma \cap S_4 \neq \emptyset\big\} \subset \bigcup_i \Big\{\gamma :\ \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \subset \partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big\}$$
$$\cup \Big\{\gamma : \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \neq \emptyset, \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \big(\mathbb{R}^{d+1} \setminus \partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\big) \neq \emptyset\Big\}$$

The curves in the first set are curves are the ones which lie on the lateral boundaries of a cylinder for a positive set of times: thus they have η measure 0 because $\mathcal{L}^{d+1}(\partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}) = 0$ for every $i \in \mathbb{N}$.

For the other set, the computation leading to (4.14) yields

$$\eta\Big(\bigcup_i \big\{\gamma : \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \neq \emptyset, \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \mathbb{R}^{d+1} \setminus \partial^l \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \neq \emptyset\big\}\Big) \leq 2(1+2\alpha) C_d \varepsilon' L^d \mathcal{H}^d(\partial\Omega),$$

where we have used (6.5). □

We now estimate the flux across the region $\partial\Omega \setminus K^{\varepsilon,\varepsilon'}_{\bar r}$.

Lemma 6.4. *It holds for $\varepsilon' \ll 1$*

$$\eta\big(\big\{\gamma : \mathrm{Graph}\,\gamma \cap (\partial\Omega \setminus K^{\varepsilon,\varepsilon'}_{\bar r})\big\}\big) < 5\varepsilon.$$

Proof. As before we observe that

$$\begin{aligned}
&\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \Big(\partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big) \neq \emptyset\Big\} \\
&\qquad \subset \Big\{\gamma : \mathrm{Graph}\,\gamma \subset \partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big\} \\
&\qquad\quad \cup \big\{\gamma : \mathrm{Graph}\,\gamma \cap S_4 \neq \emptyset\big\} \\
&\qquad\quad \cup \bigcup_{n\in\mathbb{N}} \Big\{\gamma : \mathrm{Graph}\,\gamma \cap \partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} : \mathrm{Tot.Var.}\big(\phi^{2^{-n},+} \circ \gamma\big) \geq 1\Big\} \\
&\qquad\quad \cup \bigcup_{n\in\mathbb{N}} \Big\{\gamma : \mathrm{Graph}\,\gamma \cap \partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} : \mathrm{Tot.Var.}\big(\phi^{2^{-n},-} \circ \gamma\big) \geq 1\Big\},
\end{aligned}$$

where the functions $\phi^{2^{-n},\pm}$ have been introduced in (4.3).

For the first term, as in the proof of the previous lemma, we have that (having all curves in Γ a positive length)

$$\eta\bigg(\Big\{\gamma : \gamma \subset \partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big\}\bigg) = \eta\bigg(\Big\{\gamma : \mathrm{int}\,\Big((\mathrm{id},\gamma)^{-1}\Big(\partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big)\Big) \neq \emptyset\Big\}\bigg) = 0.$$

For the second term, by Lemma 6.3, we infer

$$\big\{\gamma : \mathrm{Graph}\,\gamma \cap S_4 \neq \emptyset\big\} \leq 2(1+2\alpha) C_d \varepsilon' L^d \mathcal{H}^d(\partial\Omega).$$

Finally, to settle the last terms we argue as in Proposition 4.9: using condition (4.9b) and the fact that

$$\big|\rho(1,\mathbf{b}) \cdot \big(\nabla \phi^{2^{-n},\pm}\big)\big| \mathcal{L}^{d+1} \rightharpoonup \big|\rho(1,\mathbf{b}) \cdot \mathbf{n}\big| \mathcal{H}^d\llcorner_{\partial\Omega},$$

we deduce that

$$\big|\rho(1,\mathbf{b}) \cdot \big(\nabla \phi^{2^{-n},\pm}\big)\big| \mathcal{L}^{d+1}\Big(\mathbb{R}^{d+1} \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big) \to \int_{\mathbb{R}^{d+1} \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}} \big|\rho(1,\mathbf{b}) \cdot \mathbf{n}\big|\, \mathcal{H}^d\llcorner_{\partial\Omega}.$$

Now we have

$$\int_{\mathbb{R}^{d+1} \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}} \big|\rho(1,\mathbf{b}) \cdot \mathbf{n}\big|\, \mathcal{H}^d\llcorner_{\partial\Omega} \leq \int_{\partial\Omega \setminus K^{\varepsilon,\varepsilon'}_{\bar r}} \big|\rho(1,\mathbf{b}) \cdot \mathbf{n}\big| \mathcal{H}^d\llcorner_{\partial\Omega} < 2\varepsilon.$$

Summing up, and using for the last term Lemma 6.1, we get

$$\begin{aligned}
&\eta\bigg(\bigg\{\gamma : \mathrm{Graph}\,\gamma \cap \bigg(\partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\bigg)\bigg\}\bigg) \\
&\quad \leq \eta\big(\{\gamma : \mathrm{Graph}\,\gamma \cap S_4 \neq \emptyset\}\big) \\
&\qquad + \sum_{n\in\mathbb{N}} \eta\bigg(\bigg\{\gamma : \mathrm{Graph}\,\gamma \cap \bigg(\partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\bigg) : \mathrm{Tot.Var.}(\phi^{2^{-n},+} \circ \gamma) \geq 1\bigg\}\bigg) \\
&\qquad + \sum_{n\in\mathbb{N}} \eta\bigg(\bigg\{\gamma : \mathrm{Graph}\,\gamma \cap \bigg(\partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\bigg) : \mathrm{Tot.Var.}(\phi^{2^{-n},-} \circ \gamma) \geq 1\bigg\}\bigg) \\
&\quad \leq 2(1+2\alpha)C_d\varepsilon' L^d \mathcal{H}^d(\partial\Omega) + 2\int_{\partial\Omega\setminus K^{\varepsilon,\varepsilon'}_{\bar r}} |\rho(1,\mathbf{b})\cdot\mathbf{n}|\mathcal{H}^d \llcorner \partial\Omega \\
&\quad \leq 2(1+2\alpha)C_d\varepsilon' L^d \mathcal{H}^d(\partial\Omega) + 4\varepsilon.
\end{aligned}$$

Choosing now $\varepsilon' \ll 1$ we obtain that

$$\eta\bigg(\bigg\{\gamma : \mathrm{Graph}\,\gamma \cap \bigg(\partial\Omega \setminus \bigcup_i \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\bigg) \neq \emptyset\bigg\}\bigg) \leq 5\varepsilon.$$

Being a covering of $K^{\varepsilon,\varepsilon'}_{\bar r}$ we conclude that the statement holds. □

With the same tools we have also the following result.

Lemma 6.5. *It holds*

$$\begin{aligned}
\sum_i \eta\bigg(\bigg\{\gamma : \exists t, |s| \leq \alpha^2 r_i : \Big(\gamma(t) \in \partial\Omega \cap \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \;\wedge\; \big|\gamma(t+s) - \gamma(t) - \mathbf{b}(t_i,x_i)s\big| > 4r_i\Big)\bigg\}\bigg) \\
\leq (1+\alpha)C_d\varepsilon'(2\alpha)^{2d}\mathcal{H}^d(\partial\Omega).
\end{aligned}$$

Proof. By (half of) (6.3) we have

$$\eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \partial\Omega \cap \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \neq \emptyset, \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \not\subseteq \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\Big\}\Big) \leq \int_{\partial\mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}} \rho\big|(1,\mathbf{b})\cdot\mathbf{n}\big|\mathcal{H}^d.$$

Now, observe that

$$\begin{aligned}
&\Big\{\gamma : \exists t, |s| \leq \alpha^2 r_i\Big(\gamma(t) \in \partial\Omega \cap \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \;\wedge\; \big|\gamma(t+s) - \gamma(t) - \mathbf{b}(t_i,x_i)s\big| \geq 2r_i\Big)\Big\} \\
&\subseteq \Big\{\gamma : \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \cap \partial\Omega \cap \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \neq \emptyset, \mathrm{Graph}\,\gamma\llcorner_{J^i_\gamma} \not\subseteq \mathrm{Cyl}^{r_i,\alpha^2}_{t_i,x_i}\Big\}.
\end{aligned}$$

Summing over i we get

$$\begin{aligned}
&\sum_i \eta\Big(\Big\{\gamma : \mathrm{Graph}\,\gamma \cap \partial\Omega \cap \mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i} \neq \emptyset : \exists |s| \leq \alpha^2 r_i\big(\big|\gamma(t+s) - \gamma(t) - \mathbf{b}(t_i,x_i)s\big| > 2r_i\big)\Big\}\Big) \\
&\quad \leq \sum_i \int_{\partial\mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}} \rho\big|(1,\mathbf{b})\cdot\mathbf{n}\big|\mathcal{H}^d \leq (1+\alpha)C_d\varepsilon'(2\alpha)^{2d}\mathcal{H}^d(\partial\Omega),
\end{aligned}$$

because of (4.11). □

From Lemma 6.5 we can prove the following weak differentiability of the curves:

Corollary 6.6. *For all $\alpha > 0$ it holds*

$$\lim_{s\to 0} \eta\bigg(\bigg\{\gamma :\, \gamma(t) \in K^{\varepsilon,\varepsilon'}_{\bar r}, \bigg|\frac{\gamma(t+s) - \gamma(t)}{s} - \mathbf{b}(t,\gamma(t))\bigg| > \frac{8}{\alpha^2}\bigg\}\bigg) = 0.$$

Proof. By Lemma 4.14, we can assume that $s < \bar r$, and that there are regular cylinders in all points of $K^{\varepsilon,\varepsilon'}_{\bar r}$ with radius r such that $\frac{\alpha^2 r}{2} \leq s \leq \alpha^2 r$. Then, using these cylinders for the covering $\{\mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\}_{i=1}^{N_s}$

of $K^{\varepsilon,\varepsilon'}_{\bar r}$,

$$\Big\{\gamma:\,\gamma(t)\in K^{\varepsilon,\varepsilon'}_{\bar r},\,\bigg|\frac{\gamma(t+s)-\gamma(t)}{s}-\mathbf{b}(t,\gamma(t))\bigg|>\frac{8}{\alpha^2}\Big\}$$
$$\subset\bigcup_{i=0}^{N_s}\bigg\{\gamma:\exists t,\,\frac{\alpha^2 r_i}{2}\le s\le\alpha^2 r_i:\,\Big(\gamma(t)\in\partial\Omega\cap\mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}\ \wedge\ \big|\gamma(t+s)-\gamma(t)-\mathbf{b}(t_i,x_i)s\big|>4r_i\Big)\bigg\}.$$

Applying Lemma 6.5 and then letting $\varepsilon'\to 0$ the proof is concluded. □

We now present the following proposition which plays the role of the first part of the proof of Proposition 5.12. Recall the definition of the measures

$$\eta^i_\Omega=(\mathtt{R}^i_\Omega)_\sharp\eta,\qquad \rho_i(1,\mathbf{b})\,\mathcal{L}^{d+1}:=\int(\mathrm{id},\gamma)_\sharp\big((1,\dot\gamma)\mathcal{L}^1\big)\,\eta^i_\Omega(d\gamma),$$

given in (5.7), (5.8).

Proposition 6.7. *If $\mathtt{T}^{i,\pm}_\Omega$ are the operators defined in (5.4), then it holds*

$$(\mathtt{T}^{i,\pm}_\Omega)_\sharp\eta^i_\Omega\le\rho[(1,\mathbf{b})\cdot\mathbf{n})]^\pm\,\mathcal{H}^d\llcorner_{\partial\Omega}.$$

Proof. First of all observe that the results obtained in this section so far holds also for η^i_Ω: indeed all proofs depend only on the quantity $\rho|(1,\mathbf{b})\cdot\mathbf{n}|$, which is monotone in ρ.

By Lemma 6.4 it is enough to prove the statement in $K^{\varepsilon,\varepsilon'}_{\bar r}$, and assume that the interval of definition of γ has length at least 2τ. Hence for $r_i<\tau/\alpha^2$, up to a set of trajectories of η^i_Ω-measure of the order of ε' obtained by Lemma 6.3 when applied to $\mathbb{R}^{d+1}\setminus\mathrm{clos}\,\Omega$, all trajectories of η^i_Ω starting from $K^{\varepsilon,\varepsilon'}_{\bar r}\cap\mathrm{Cyl}^{r,2\alpha^2}_{t,x}$ exit the cylinder by crossing one of the flat bases. In particular we deduce that up to $\mathcal{O}(\varepsilon+\varepsilon')$ trajectories, $(\mathtt{T}^{i,\pm}_\Omega)_\sharp\eta^i_\Omega$ is concentrated on $K^{\varepsilon,\varepsilon'}_{\bar r}\cap\{(1,\mathbf{b})\cdot\mathbf{n}\gtrless 0\}$. Hence $(\mathtt{T}^{i,\pm}_\Omega)_\sharp\eta^i_\Omega$ are orthogonal.

Since it holds

$$0\le\int\rho_i\Big(\big|(1,\mathbf{b})\cdot\nabla_{t,x}\phi^{\delta,-}\big|-(1,\mathbf{b})\cdot\nabla_{t,x}\phi^{\delta,-}\Big)\,\mathcal{L}^{d+1}\le\int\rho\Big(\big|(1,\mathbf{b})\cdot\nabla_{t,x}\phi^{\delta,-}\big|-(1,\mathbf{b})\cdot\nabla_{t,x}\phi^{\delta,-}\Big)\,\mathcal{L}^{d+1},$$

using the weak convergence of $\rho(1,\mathbf{b})\cdot\nabla_{t,x}\phi^{\delta,-}$ and $\rho|(1,\mathbf{b})\cdot\nabla_{t,x}\phi^{\delta,-}|$ together with the fact $\rho_i\le\rho$ we obtain the statement. □

In particular the behavior (entering/exiting) of trajectories which crosses Ω does not depend on the particular characteristic, but only on the sign of $(1,\mathbf{b})\cdot\mathbf{n}$. It follows from the trace analysis that the same property of BD vector field holds also for proper sets.

Theorem 6.8. *If Ω is a proper set, the restriction operator $\mathtt{R}_\Omega$ maps a Lagrangian representation of $\rho(1,\mathbf{b})$ to a Lagrangian representation of $\rho(1,\mathbf{b})\llcorner_\Omega$.*

Proof. Using Proposition 6.7 we can define the sets

$$A^\pm=\big\{(t,x)\in\partial\Omega:(1,\mathbf{b})\cdot\mathbf{n}(t,x)\gtrless 0\big\}.$$

Now it is sufficient to repeat the proof of Proposition 5.12. □

Corollary 6.9. *A Lagrangian η of $\rho(1,\mathbf{b})\mathcal{L}^{d+1}$ is concentrated on the set*

$$\bigcup_{N\in\mathbb{N}}\bigg\{\gamma:(\mathrm{id},\gamma)^{-1}(\Omega)=\bigcup_{i=1}^N\big(t^{i,-}_\gamma,t^{i,+}_\gamma\big),\ (\mathrm{id},\gamma)^{-1}(\mathrm{clos}\,\Omega)=\bigcup_{i=1}^N\big[t^{i,-}_\gamma,t^{i,+}_\gamma\big]\ \textit{with}\ t^{i,+}_\gamma<t^{i+1,-}_\gamma\bigg\}.$$

Moreover, if η^ε is a Lagrangian representation of $\rho(1,\mathbf{b})\mathcal{L}^{d+1}\llcorner_{\Omega^\varepsilon}$, then

$$\lim_{\varepsilon\to 0}\eta^\varepsilon\big(\{\gamma:(\mathrm{id},\gamma)^{-1}(\Omega)\text{ is not an interval}\}\big)=0.$$

Proof. For the first part of the statement, observe that by the absolute convergence of the series $\sum(\mathtt{T}^{i,\pm}_\Omega)_\sharp\eta$ it follows that η is concentrated on the set

$$\bigcup_{N\in\mathbb{N}}\bigg\{\gamma:(\mathrm{id},\gamma)^{-1}(\Omega)=\bigcup_{i=1}^N\big(t^{i,-}_\gamma,t^{i,+}_\gamma\big)\bigg\}.$$

On the other hand, since the set of curves which lie on $\partial\Omega$ for a positive amount of time is negligible, it follows that

$$(\mathrm{id},\gamma)^{-1}(\operatorname{clos}\Omega) = \bigcup_{i=1}^{N} \big[t_\gamma^{i,-}, t_\gamma^{i,+}\big]$$

for η-a.e. curve such that $(\mathrm{id},\gamma)^{-1}(\Omega)$ is made of finitely many open intervals. Finally, by Corollary 6.6 the set of curves which have $t_\gamma^{i,-} = t_\gamma^{i+1,-}$ is negligible.

The second part of the statement follows by observing that if a curve γ is such that $\operatorname{Graph}\gamma \in \Omega^\varepsilon$ and $(\mathrm{id},\gamma)^{-1}(\Omega)$ is not an interval, then up to a measure of order ε' it must re-enter in Ω (re-exit from Ω) in the same cylinder $\mathrm{Cyl}^{r_i,2\alpha^2}_{t_i,x_i}$ where it just exited (entered). By Proposition 6.7, this is controlled by the entering (exiting) flow in a neighborhood of $K^{\varepsilon,\varepsilon'}_{\bar r} \cap \{(1,\mathbf{b}) > 0\}$, this can be made arbitrarily small as $\varepsilon \to 0$. ☐

To end this section we present the following

Proposition 6.10. *Let $\Omega \subset \mathbb{R}^{d+1}$ be a proper set and $N \subset \Gamma$ a Borel set. It holds*

$$\eta\Big(\Big\{\gamma : \exists i \text{ s.t. } \mathtt{R}^i_\Omega\gamma = \gamma\llcorner_{(t_\gamma^{i-},t_\gamma^{i+})} \in N\Big\}\Big) \leq (\mathtt{R}_\Omega)_\sharp\eta(N).$$

Proof. Let $\tilde N$ be the set given by

$$\tilde N := \Big\{(\gamma,i) \in \Gamma \times \mathbb{N} : \mathtt{R}^i_\Omega\gamma = \gamma\llcorner_{(t_\gamma^{i-},t_\gamma^{i+})} \in N\Big\},$$

which is a Borel set because the map $\mathtt{R}^i_\Omega$ is Borel (see Lemma 5.7). Let

$$\pi_1(\tilde N) \ni \gamma \mapsto i(\gamma)$$

be a Borel selection which exists because $\tilde N$ is countable union of Borel graphs. We estimate by using the definition of $\mathtt{R}_\Omega$

$$\begin{aligned}
(\mathtt{R}_\Omega)_\sharp\eta(N) &= \sum_j (\mathtt{R}^j_\Omega)_\sharp\eta(N) \\
&\geq \sum_j (\mathtt{R}^j_\Omega)_\sharp\eta(\{\gamma : i(\gamma) = j\}) \\
&= \sum_j \eta(\{\gamma : i(\gamma) = j\}) \\
&= \eta(\pi_1(\tilde N)).
\end{aligned}$$

☐

Together with Corollary 6.9 we deduce

Corollary 6.11. *For all $N \subset \Gamma$ is holds*

$$\lim_{\varepsilon\searrow 0} (\mathtt{R}_{\Omega^\varepsilon})_\sharp\eta\big(\{\gamma : \exists i \text{ s.t. } \mathtt{R}^i_{\Omega^\varepsilon}\gamma \in N\}\big) = (\mathtt{R}_\Omega)_\sharp\eta(N). \tag{6.6}$$

Proof. Just observe that the equality in (6.6) above holds when $(\mathrm{id},\gamma)^{-1}(\Omega)$ is a single interval, and apply Corollary 6.9. ☐

Part 2
Cylinders of approximate flow and untangling of trajectories

This part deals with the main result of the paper:

give a local condition on the vector field in order to construct a partition of $\mathbb{R}^{d+1}$ into disjoint trajectories such that η-a.e. γ is a subset of these curves.

We call this property *untangling*: it is stronger than uniqueness for initial data at some given time $\bar{t}$ (even if it can be deduced from this uniqueness by the analysis below), because it implies that no crossings occur on "bad" sets (i.e. non rectifiable, Cantor-like, in general sets on which one cannot assign meaningful initial data).

The condition we give is quite general, and can be adapted to the particular case under consideration: it can be resumed by saying that we control the measure of trajectories entering and exiting from arbitrarily small cylinders around η-a.e. trajectory γ in terms of the $\mathcal{L}^d$-measure of their base. This yields a control of the amount of trajectories which bifurcate in the future or in the past from another given trajectory, and it can be nicely expressed in terms of transference plans.

A duality result yields that a control on the flow across the boundary of these cylinders implies an estimate of the amount of trajectories which have a common point but are not subsets of a unique trajectory. This leads to the introduction of the *untangling functional*, which measures the minimal amount of trajectories one has to remove in order to obtain a disjoint set of trajectories such that η-a.e. γ is a subset of these. This functional turns out to be subadditive, allowing a natural condition in order to extend a local estimate to a global one.

The last part shows that in the case of untangling the structure of the representation allows the complete description of the disintegration of the PDE, in particular the computation of the chain rule.

7. Cylinders of approximate flow

7.1. **Cylinders of approximate flow and transference plans.** Consider a proper set $\Omega \subset \mathbb{R}^{d+1}$, and let Ω^ε be the pertubed set constructed in Theorem 4.16. For convenience, in the first part of this section we will drop the index ε and refer to Ω^ε directly as Ω. Furthermore, η will denote a Lagrangian representation of $\mathrm{div}(\rho(1,\mathbf{b})) = \mu$ in Ω (which can be taken as the restriction of a Lagrangian representation in $\mathbb{R}^{d+1}$, in view of Theorem 6.8).

Recall that the set S_1 is defined in (4.13), so that essentially all inflow and outflow of $\rho(1,\mathbf{b})$ are occurring on open sets which are contained in finitely many time-flat hyperplanes $\{t = t_i\}$. We can assume without loss of generality that $\mathtt{p}_t(S_1) \subset \{\{t = t_i\}$ is locally proper$\}$. Define now

$$\eta^{\mathrm{in}} := \int_{S_1} \eta^{\mathrm{in}}_z \rho(z)\, \mathcal{H}^d(dz) = \eta \llcorner_{\{\mathrm{Graph}\, \gamma \cap S_1 \neq \emptyset\}},$$

according to (3.3).

We assume that the following.

Assumption 7.1. There are constants $\mathtt{M}, \varpi > 0$ and a family of functions $\{\phi^\ell_\gamma\}_{\ell>0,\gamma\in\Gamma}$ such that:

1. for every $\gamma \in \Gamma, \ell \in \mathbb{R}^+$, the function $\phi^\ell_\gamma \colon [t^-_\gamma, t^+_\gamma] \times \mathbb{R}^d \to [0,1]$ is Lipschitz;
2. for $t \in [t^-_\gamma, t^+_\gamma]$, $x \in \mathbb{R}^d$
$$\mathbb{1}_{\gamma(t)+B^d_{\ell/\mathtt{M}}(0)}(x) \leq \phi^\ell_\gamma(t,x) \leq \mathbb{1}_{\gamma(t)+B^d_{\mathtt{M}\ell}(0)}(x);$$
3. it holds
$$\int \left\{ \frac{1}{\sigma(\phi^\ell_\gamma(t^-_\gamma))} \int_{t^-_\gamma}^{t^+_\gamma} \left[\int \rho(t) \big| (1,\mathbf{b}) \cdot \nabla \phi^\ell_\gamma(t) \big| \, \mathcal{L}^d \right] dt \right\} \eta^{\mathrm{in}}(d\gamma) \leq \varpi, \tag{7.1}$$
where
$$\sigma(f(t)) = \int f(t,x)\rho(t,x)\, \mathcal{L}^d(dx), \tag{7.2}$$
for every $t \in \mathtt{p}_1(S_1)$.

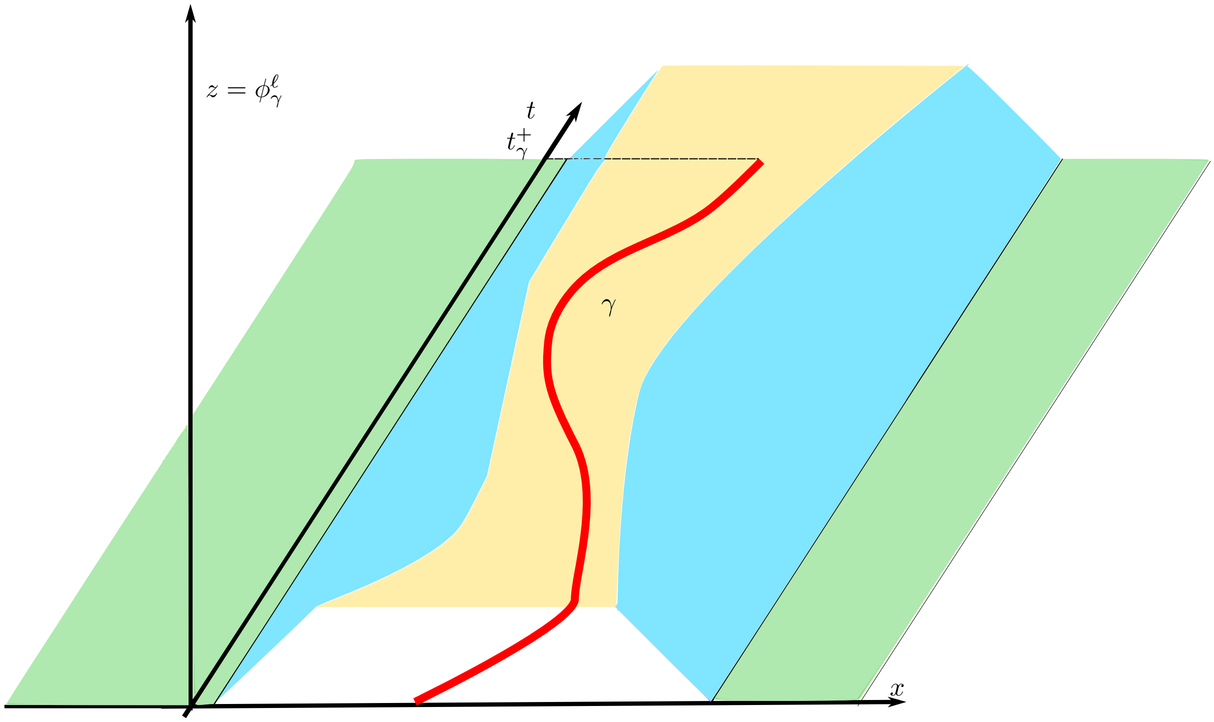


Figure 7. A cylinder of approximate flow ϕ^ℓ_γ.

From now onwards we will often refer to the family of functions $\{\phi^\ell_\gamma\}_{\ell>0,\gamma\in\Gamma}$ as *cylinders of approximate flow*: indeed, if γ is a characteristic of the vector field $\mathbf{b}$, the function ϕ^ℓ_γ can be thought as generalized, smoothed cylinder centered at γ (see Fig. 7). In particular, Point (3) is saying that the flow through the "lateral boundary of the cylinder" is controlled by the quantity ϖ.

Introduce the set

$$W := W_1 \cup W_2 \subset \Gamma \times \Gamma$$

where W_1 is the open set

$$W_1 := \big\{(\gamma,\gamma') : \mathrm{Graph}\,\gamma \cap \mathrm{Graph}\,\gamma' = \emptyset\big\},$$

while W_2 is the closed set

$$W_2 := \Big\{(\gamma,\gamma') : \mathrm{Graph}\,\gamma \cap \mathrm{Graph}\,\gamma' = \mathrm{Graph}\,\big(\gamma\llcorner_{[\max\{t^-_\gamma,t^-_{\gamma'}\},\min\{t^+_\gamma,t^+_{\gamma'}\}]}\big)\Big\}.$$

Thus the set W is a Borel set (we recall that $\mathrm{Graph}\,\gamma$ is the set of points $(t,\gamma(t))$ for t in the closed interval $[t^-_\gamma, t^+_\gamma]$, see (3.4)).

Proposition 7.2. *Under Assumption 7.1, it holds*

$$\int_{S_1} \eta^{\mathrm{in}}_z \otimes \eta^{\mathrm{in}}_z(\Gamma^2 \setminus W_2)\rho(z)\,\mathcal{H}^d(dz) \le \varpi.$$

Proof. We split the proof in several steps.
Step 1. For fixed $\ell > 0$ and $\gamma \in \Gamma$ we introduce the following set

$$E^\ell_\gamma := \big\{\gamma' : \mathrm{Graph}\,\gamma'\llcorner_{[t^-_\gamma,t^+_\gamma]} \not\subseteq \mathrm{supp}\,\phi^\ell_\gamma\big\} \subset \Gamma$$

and consider the functional

$$\Phi^\ell_{\mathrm{exit}}(\gamma) := \int_{S_1\cap\{t=t^-_\gamma\}} \eta^{\mathrm{in}}_{z'}\big(E^\ell_\gamma\big)\phi^\ell_\gamma(t^-_\gamma, z')\rho(z')\,\mathcal{H}^d(dz').$$

This functional computes the weighted amount of curves γ' starting inside $\mathrm{supp}\,\phi^\ell_\gamma \cap S_1$ and exiting from the cylinder.

Noticing that

$$\mathrm{Tot.Var.}\big(\phi^\ell_\gamma \circ \gamma'\llcorner_{[t^-_\gamma,t^+_\gamma]}\big) \geq \phi^\ell_\gamma(z') \qquad \text{when } \gamma'(t^-_\gamma = t^-_{\gamma'}) = z',\ \mathrm{Graph}\,\gamma'\llcorner_{[t^-_\gamma,t^+_\gamma]} \not\subseteq \mathrm{supp}\,\phi^\ell_\gamma,$$

we have

$$\begin{aligned}
\Phi^\ell_{\mathrm{exit}}(\gamma) &= \int_{S_1\cap\{t=t^-_\gamma\}} \eta^{\mathrm{in}}_{z'}\big(E^\ell_\gamma\big)\phi^\ell_\gamma(z')\rho(z')\,\mathcal{H}^d(dz') \\
&= \int_{\{\gamma':\,t^-_\gamma=t^-_{\gamma'},\,\mathrm{Graph}\,\gamma'\llcorner_{[t^-_\gamma,t^+_\gamma]}\not\subseteq \mathrm{supp}\,\phi^\ell_\gamma,\phi^\ell_\gamma(\gamma'(t^-_{\gamma'}))>0\}} \phi^\ell_\gamma\big(t^-_{\gamma'},\gamma'(t^-_{\gamma'})\big)\,\eta^{\mathrm{in}}(d\gamma') \\
&\leq \int_{\{\gamma':\,t^-_\gamma=t^-_{\gamma'},\,\mathrm{Graph}\,\gamma'\llcorner_{[t^-_\gamma,t^+_\gamma]}\not\subseteq \mathrm{supp}\,\phi^\ell_\gamma,\phi^\ell_\gamma(\gamma'(t^-_{\gamma'}))>0\}} \mathrm{Tot.Var.}\big(\phi^\ell_\gamma\circ\gamma'\llcorner_{[t^-_\gamma,t^+_\gamma]}\big)\,\eta^{\mathrm{in}}(d\gamma') \\
&\leq \int \mathrm{Tot.Var.}\big(\phi^\ell_\gamma\circ\gamma'\llcorner_{[t^-_\gamma,t^+_\gamma]}\big)\,\eta(d\gamma') \\
&\leq \int_{t^-_\gamma}^{t^+_\gamma} \bigg[\int \rho(t,x)\big|(1,\mathbf{b})(t,x)\cdot\nabla_{t,x}\phi^\ell_\gamma(t,x)\big|\,\mathcal{L}^d(dx)\bigg]\,dt,
\end{aligned}$$

so that using Point (3), we deduce

$$\int_\Gamma \frac{1}{\sigma(\phi^\ell_\gamma(t^-_\gamma))}\Phi^\ell_{\mathrm{exit}}(\gamma)\,\eta^{\mathrm{in}}(d\gamma) \leq \varpi.$$

Step 2. Consider now a sequence $\ell_i \to 0$ such that

$$\phi^{\ell_i}_\gamma \geq \phi^{\ell_{i+1}}_\gamma. \tag{7.3}$$

Due to Point (2), Assumption 7.1 this can be achieved if

$$\ell_{i+1} \leq \frac{\ell_i}{\mathtt{M}^2},$$

because with this choice

$$\mathrm{supp}\,\phi^{\ell_{i+1}}_\gamma(t) \subset \gamma(t) + B^d_{\mathtt{M}\ell_{i+1}} \subset \gamma(t) + B^d_{\ell_i/\mathtt{M}} \subset \big\{\phi^{\ell_i}_\gamma(t) = 1\big\}. \tag{7.4}$$

Step 3. Thanks to the choice of the sequence ℓ_i in Step 2, we can estimate for $j < i$

$$\begin{aligned}
\varpi &\geq \int \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))}\Phi^{\ell_j}_{\mathrm{exit}}(\gamma)\,\eta^{\mathrm{in}}(d\gamma) \\
&= \int \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))}\bigg\{\int_{S_1\cap\{t=t^-_\gamma\}} \eta^{\mathrm{in}}_{z'}\big(E^{\ell_j}_\gamma\big)\phi^{\ell_j}_\gamma(z')\rho(z')\,\mathcal{H}^d(dz')\bigg\}\,\eta^{\mathrm{in}}(d\gamma) \\
(E^{\ell_i}_\gamma \subset E^{\ell_j}_\gamma)\ &\geq \int \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))}\bigg\{\int_{S_1\cap\{t=t^-_\gamma\}} \eta^{\mathrm{in}}_{z'}\big(E^{\ell_i}_\gamma\big)\phi^{\ell_j}_\gamma(z')\rho(z')\,\mathcal{H}^d(dz')\bigg\}\,\eta^{\mathrm{in}}(d\gamma).
\end{aligned}$$

Now, for fixed i, we pass to the limit as $j \to +\infty$ and we observe that

$$\frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))}\int_{S_1\cap\{t=t^-_\gamma\}} \eta^{\mathrm{in}}_z\phi^{\ell_j}_\gamma(z)\rho(z)\,\mathcal{H}^d(dz) \rightharpoonup \eta^{\mathrm{in}}_{\gamma(t^-_\gamma)} \qquad \text{weakly}^*$$

in duality w.r.t. continuous, bounded functions for η^{in}-a.e. γ. This follows from the fact that $\rho\mathcal{H}^d$-a.e. $z' \in S_1$ is a Lebesgue point for the map $z' \mapsto \eta^{\mathrm{in}}_{z'}$ and the set of γ starting in a negligible set in S_1 is η^{in} negligible. Notice that for every $i \in \mathbb{N}$ the set $E^{\ell_i}_\gamma$ is open, so that thanks to the l.s.c. of the weak convergence on open sets, we have

$$\eta^{\mathrm{in}}_{\gamma(t^-_\gamma)}\big(E^{\ell_i}_\gamma\big) \leq \liminf_{j\to\infty} \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))}\int_{S_1\cap\{t=t^-_\gamma\}} \phi^{\ell_j}_\gamma(z')\eta_{z'}\big(E^{\ell_i}_\gamma\big)\rho(z')\,\mathcal{H}^d(dz').$$

Step 4. Using Fatou's Lemma, we conclude that

$$\begin{aligned}
\varpi &\geq \liminf_{j\to\infty} \int_{S_1} \bigg\{ \int \frac{1}{\sigma(\phi_\gamma^{\ell_j}(t_\gamma^-))} \bigg[\int_{\operatorname{supp}\phi_\gamma^{\ell_j}(t_\gamma^-)} \phi_\gamma^{\ell_j}(z)\eta_z(E_\gamma^{\ell_i})\rho(z)\,\mathcal{H}^d(dz) \bigg] \eta_{z'}(d\gamma) \bigg\} \rho(z')\,\mathcal{H}^d(dz') \\
&\geq \int_{S_1} \bigg\{ \int \liminf_j \frac{1}{\sigma(\phi_\gamma^{\ell_j}(t_\gamma^-))} \bigg[\int_{\operatorname{supp}\phi_\gamma^{\ell_j}(t_\gamma^-)} \phi_\gamma^{\ell_j}(z)\eta_z(E_\gamma^{\ell_i})\rho(z)\,\mathcal{H}^d(dz) \bigg] \eta_{z'}(d\gamma) \bigg\} \rho(z')\,\mathcal{H}^d(dz') \\
&\geq \int_{S_1} \bigg\{ \int \eta_{z'}(E_\gamma^{\ell_i})\,\eta_{z'}(d\gamma) \bigg\} \rho(z')\,\mathcal{H}^d(dz') \\
&= \int_{S_1} \eta_{z'} \otimes \eta_{z'} \big(\{(\gamma,\gamma') : \gamma' \in E_\gamma^{\ell_i}\} \big) \rho(z')\,\mathcal{H}^d(dz').
\end{aligned} \tag{7.5}$$

Observe now that when $i \to \infty$

$$\{(\gamma,\gamma') : \gamma' \in E_\gamma^{\ell_i}\} \nearrow \mathit{\Gamma}^2 \setminus W_2.$$

By the Monotone Convergence Theorem, we then conclude

$$\int_{S_1} \eta_{z'} \otimes \eta_{z'}(\mathit{\Gamma}^2 \setminus W_2)\rho(z')\,\mathcal{H}^d(dz') = \lim_i \int_{S_1} \eta_{z'} \otimes \eta_{z'}\big(\{(\gamma,\gamma') : \gamma' \in E_\gamma^{\ell_i}\}\big)\rho(z')\,\mathcal{H}^d(dz') \leq \varpi,$$

which concludes the proof. $\square$

To analyze the trajectories which are entering into the cylinder ϕ_ℓ^γ, we have to introduce a new object. Let $\pi \in \mathsf{Adm}(\eta^{\mathrm{in}}, \eta)$ be an admissible plan between the measures η^{in} and η: this means that

$$(\mathtt{p}_1)_\sharp \pi = g_1 \eta^{\mathrm{in}}, \qquad (\mathtt{p}_2)_\sharp \pi = g_2 \eta,$$

with $0 \leq g_1, g_2 \leq 1$ are Borel functions. Observe that by disintegration we have

$$\pi = \int \pi_\gamma\, \eta^{\mathrm{in}}(d\gamma) = \int_{S_1} \bigg[\int \pi_\gamma\, \eta_z^{\mathrm{in}}(d\gamma) \bigg] \rho(z)\,\mathcal{H}^d(dz),$$

with $\|\pi_\gamma\| = g_1(\gamma)$, and similarly for the disintegration w.r.t. the second marginal η.

The following proposition is the analogue of Proposition 7.2 for the plan π.

Proposition 7.3. *Under Assumption 7.1, it holds*

$$\int \bigg\{ \int \pi_{\gamma'}\Big(\Big\{ (\gamma',\gamma'') : \gamma''(t_{\gamma''}^-) \notin \operatorname{Graph}\gamma,\, (\gamma' \llcorner_{[t_\gamma^-, t_\gamma^+]}, \gamma'' \llcorner_{[t_\gamma^-, t_\gamma^+]}) \in \mathit{\Gamma}^2 \setminus W_1 \Big\} \Big)\, \eta_z^{\mathrm{in}} \otimes \eta_z^{\mathrm{in}}(d\gamma d\gamma') \bigg\} \rho(z)\,\mathcal{H}^d(dz) \leq \varpi.$$

Proof. We split the proof in several steps.

Step 1. For fixed $\ell > 0$ and $\gamma \in \mathit{\Gamma}$ we introduce the following set

$$A_\gamma^\ell := \Big\{ (\gamma',\gamma'') : \phi_\gamma^\ell(\gamma''(\max\{t_{\gamma''}^-, t_\gamma^-\})) = 0,\ \big(\gamma' \llcorner_{[t_\gamma^-, t_\gamma^+]}, \gamma'' \llcorner_{[t_\gamma^-, t_\gamma^+]}\big) \in \mathit{\Gamma}^2 \setminus W_1 \Big\}, \tag{7.6}$$

and consider the functional

$$\Phi_{\mathrm{enter}}^\ell(\gamma) := \int_{S_1 \cap \{t = t_\gamma^-\}} \bigg[\int \pi_{\gamma'}(A_\gamma^\ell)\, \eta_{z'}^{\mathrm{in}}(d\gamma') \bigg] \phi_\gamma^\ell(z')\rho(z')\,\mathcal{H}^d(dz').$$

This integral computes the weighted amount of curves γ'' starting outside the cylinder ϕ_γ^ℓ and touching a curve γ' which starts inside the cylinder in the time interval $[t_\gamma^-, t_\gamma^+]$.

We observe that for every $(\gamma',\gamma'') \in A_\gamma^\ell$ it holds

$$\mathrm{Tot.Var.}(\phi_\gamma^\ell \circ \gamma' \llcorner_{[t_\gamma^-, t_\gamma^+]}) + \mathrm{Tot.Var.}(\phi_\gamma^\ell \circ \gamma'' \llcorner_{[t_\gamma^-, t_\gamma^+]}) \geq \phi_\gamma^\ell(z'), \quad \text{when} \quad \gamma'(t_{\gamma'}^- = t_\gamma^-) = z'. \tag{7.7}$$

Then we have, by integration,

$$\begin{aligned}
\Phi_{\mathrm{enter}}^\ell(\gamma) &= \int_{S_1 \cap \{t = t_\gamma^-\}} \bigg[\int \pi_{\gamma'}(A_\gamma^\ell)\, \eta_{z'}^{\mathrm{in}}(d\gamma') \bigg] \phi_\gamma^\ell(z')\rho(z')\,\mathcal{H}^d(dz') \\
&= \int_{A_\gamma^\ell \cap \{(\gamma',\gamma'') :\, t_\gamma^- = t_{\gamma'}^-\}} \phi_\gamma^\ell(\gamma'(t_{\gamma'}^-))\, \pi(d\gamma' d\gamma'') \\
&= \int_{A_\gamma^\ell \cap \{(\gamma',\gamma'') :\, t_\gamma^- = t_{\gamma'}^-,\, \phi_\gamma^\ell(\gamma'(t_{\gamma'}^-)) > 0\}} \phi_\gamma^\ell(\gamma'(t_{\gamma'}^-))\, \pi(d\gamma' d\gamma'') \boxed{\leq}
\end{aligned}$$

so that, taking into account (7.7), we get

$$\begin{aligned}
\boxed{\leq} & \int_{A^\ell_\gamma \cap \{(\gamma',\gamma'') :\, t^-_\gamma = t^-_{\gamma'},\, \phi^\ell_\gamma(\gamma'(t^-_{\gamma'}))>0\}} \Big[\mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]}) + \mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma'' \llcorner_{[t^-_\gamma, t^+_\gamma]}) \Big] \, \pi(d\gamma' d\gamma'') \\
= & \int_{\{\gamma' : t^-_\gamma = t^-_{\gamma'}, \phi^\ell_\gamma(\gamma'(t^-_{\gamma'}))>0\}} \pi_{\gamma'}\big(A^\ell_\gamma\big) \mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]}) \, \eta^{\mathrm{in}}(d\gamma') \\
& + \int_{\{\gamma'' :\, \phi^\ell_\gamma(\gamma''(\max\{t^-_\gamma, t^-_{\gamma''}\}))=0\}} \pi_{\gamma''}\big(A^\ell_\gamma \cap \{\gamma' : t^-_\gamma = t^-_{\gamma'}, \phi^\ell_\gamma(\gamma'(t^-_{\gamma'}))>0\}\big) \mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma'' \llcorner_{[t^-_\gamma, t^+_\gamma]}) \, \eta(d\gamma'') \\
\leq & \int_{\{\gamma' : t^-_\gamma = t^-_{\gamma'}, \phi^\ell_\gamma(\gamma'(t^-_{\gamma'}))>0\}} \mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]}) \, \eta^{\mathrm{in}}(d\gamma') \\
& + \int_{\{\gamma'' :\, \phi^\ell_\gamma(\gamma''(\max\{t^-_\gamma, t^-_{\gamma''}\}))=0\}} \mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma'' \llcorner_{[t^-_\gamma, t^+_\gamma]}) \, \eta(d\gamma'') \\
\leq & \int \mathrm{Tot.Var.}(\phi^\ell_\gamma \circ \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]}) \, \eta(d\gamma') \\
\leq & \int_{t^-_\gamma}^{t^+_\gamma} \bigg[\int \rho(t,x) \big| (1, \mathbf{b}(t,x)) \cdot \nabla_{t,x} \phi^\ell_\gamma(t,x) \big| \, \mathcal{L}^d(dx) \bigg] \, dt.
\end{aligned}$$

Integrating in γ and using Point (3), we deduce

$$\int \frac{1}{\sigma(\phi^\ell_\gamma(t^-_\gamma))} \Phi^\ell_{\mathrm{enter}}(\gamma) \, \eta^{\mathrm{in}}(d\gamma) \leq \varpi. \tag{7.8}$$

Step 2. Consider now a sequence $\ell_i \to 0$ such that

$$\{\phi^{\ell_i}_\gamma < a\} \subset \{\phi^{\ell_j}_\gamma = 0\} \tag{7.9}$$

for every $i < j$. For instance, the same choice as in *Step 2* of Proposition 7.2 is sufficient for $a = 1$, thanks to (7.4).

Step 3. We now pass to the limit. By (7.8), we have

$$\begin{aligned}
\varpi & \geq \int \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))} \Phi^{\ell_j}_{\mathrm{enter}}(\gamma) \, \eta^{\mathrm{in}}(d\gamma) \\
& = \int \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))} \bigg\{ \int_{S_1 \cap \{t = t^-_\gamma\}} \bigg[\int \pi_{\gamma'}(A^{\ell_j}_\gamma) \, \eta^{\mathrm{in}}_{z'}(d\gamma') \bigg] \phi^{\ell_j}_\gamma(z') \rho(z') \, \mathcal{H}^d(dz') \bigg\} \, \eta^{\mathrm{in}}(d\gamma).
\end{aligned}$$

where we recall the set A^ℓ_γ is defined in (7.6) as

$$A^\ell_\gamma = \Big\{ (\gamma', \gamma'') : \phi^\ell_\gamma(\gamma''(\max\{t^-_{\gamma''}, t^-_\gamma\})) = 0, \ \big(\gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]}, \gamma'' \llcorner_{[t^-_\gamma, t^+_\gamma]}\big) \in \Gamma^2 \setminus W_1 \Big\}.$$

To overcome the difficulty given by the fact that A^ℓ_γ is not open, we take into account *Step 2* and define the open set

$$A^{\ell,a}_\gamma := \Big\{ (\gamma', \gamma'') : \phi^\ell_\gamma(\gamma''(\max\{t^-_{\gamma''}, t^-_\gamma\})) < a, \ \big(\gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]}, \gamma'' \llcorner_{[t^-_\gamma, t^+_\gamma]}\big) \in \Gamma^2 \setminus W_1 \Big\}.$$

Notice so that, thanks to the (7.4), $A^{\ell_i,a}_\gamma \subset A^{\ell_j}_\gamma$ for $i < j$ and hence

$$\begin{aligned}
& \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))} \int_{S_1 \cap \{t = t^-_\gamma\}} \bigg[\int \pi_{\gamma'}(A^{\ell_j}_\gamma) \, \eta^{\mathrm{in}}_{z'}(d\gamma') \bigg] \phi^{\ell_j}_\gamma(z') \rho(z') \, \mathcal{H}^d(dz') \\
& \geq \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))} \int_{S_1 \cap \{t = t^-_\gamma\}} \bigg[\int \pi_{\gamma'}(A^{\ell_i,a}_\gamma) \, \eta^{\mathrm{in}}_{z'}(d\gamma') \bigg] \phi^{\ell_j}_\gamma(z') \rho(z') \, \mathcal{H}^d(dz') \\
& = \frac{1}{\sigma(\phi^{\ell_j}_\gamma(t^-_\gamma))} \int_{S_1 \cap \{t = t^-_\gamma\}} \bigg\{ \int \Big[\underbrace{(\pi \llcorner_{\Gamma^2 \setminus W_1(\gamma)})_{\gamma'} \big(\{\gamma'' : \, \phi^{\ell_i}_\gamma(\gamma''(\max\{t^-_{\gamma''}, t^-_\gamma\})) < a\} \big)}_{\mathbf{I}(\gamma, \gamma', \ell_i)} \Big] \, \eta^{\mathrm{in}}_{z'}(d\gamma') \bigg\} \phi^{\ell_j}_\gamma(z') \rho(z') \, \mathcal{H}^d(dz'),
\end{aligned}$$

where

$$\Gamma^2 \setminus W_1(\gamma) = \Big\{(\gamma',\gamma''): \big(\gamma' \llcorner_{[t_\gamma^-,t_\gamma^+]}, \gamma'' \llcorner_{[t_\gamma^-,t_\gamma^+]}\big) \in \Gamma^2 \setminus W_1\Big\}.$$

Step 4. Define for $t_1 < t_2$ the set

$$\begin{aligned}\Gamma^2 \setminus W_1(t_1,t_2) &= \Big\{(\gamma',\gamma''): \big(\gamma' \llcorner_{[t_1,t_2]}, \gamma'' \llcorner_{[t_1,t_2]}\big) \in \Gamma^2 \setminus W_1\Big\} \\ &= \Big\{(\gamma',\gamma''): \mathrm{Graph}\,\gamma' \llcorner_{[t_1,t_2]} \cap \mathrm{Graph}\,\gamma'' \llcorner_{[t_1,t_2]} \neq \emptyset\Big\},\end{aligned}$$

and accordingly let

$$\mathbf{I}(t_1,t_2,\gamma',\ell_i) = (\pi \llcorner_{\Gamma^2 \setminus W_1(t_1,t_2)})_{\gamma'}\big(\big\{\gamma'' : \phi_\gamma^{\ell_i}(\gamma''(\max\{t^-_{\gamma''}, t^-_\gamma\})) < a\big\}\big).$$

Now $\rho\mathcal{H}^d$-a.e. $z' \in S_1$ is a Lebesgue point for the map

$$z' \mapsto \int \big[(\pi \llcorner_{\Gamma^2 \setminus W_1(t_1,t_2)})_{\gamma'}\big]\, \eta^{\mathrm{in}}_{z'}(d\gamma'),$$

w.r.t. the weak* topology, and hence, arguing as in Proposition 7.2, passing to the limit in j and using the l.s.c. on open sets (i.e. $\{\gamma'' : \phi_\gamma^{\ell_i}(\gamma''(\max\{t^-_{\gamma''}, t^-_\gamma\})) < a\}$) we deduce

$$\begin{aligned}&\int I(t_1,t_2,\gamma',\ell_i)\, \eta^{\mathrm{in}}_{\gamma(t_\gamma^-)}(d\gamma') \\ &\quad \leq \liminf_{j\to+\infty} \frac{1}{\sigma(\phi_\gamma^{\ell_j}(t_\gamma^-))} \int_{S_1 \cap \{t=t_\gamma^-\}} \bigg[\int \mathbf{I}(t_1,t_2,\gamma',\ell_i)\, \eta^{\mathrm{in}}_{z'}(d\gamma')\bigg] \phi_\gamma^{\ell_j}(z')\rho(z')\,\mathcal{H}^d(dz')\end{aligned}$$

for η^{in}-a.e. γ.

Step 5. Take a partition of a set where η^{in} is concentrated into finitely many disjoint sets $\{A^{\mathrm{in}}_{k,n}\}_{n=1}^{N_k}$ so that

$$A^{\mathrm{in}}_{k,n} \subset \Big\{\gamma \in \Gamma : t_n^- - 2^{-k} < t_\gamma^- < t_n^-, t_n^+ \leq t_\gamma^+ \leq t_n^+ + 2^{-k}\Big\}$$

and a set $A^{\mathrm{in}}_{k,0}$ whose measure is arbitrarily small for $k \to \infty$. *Step 3* above gives

$$\begin{aligned}\varpi &\geq \int_\Gamma \frac{1}{\sigma(\phi_\gamma^{\ell_j}(t_\gamma^-))} \int_{S_1\cap\{t=t_\gamma^-\}} \bigg\{\int \mathbf{I}(\gamma,\gamma',\ell_i)\, \eta^{\mathrm{in}}_{z'}(d\gamma')\bigg\} \phi_\gamma^{\ell_j}(z')\rho(z')\,\mathcal{H}^d(dz')\bigg\}\,\eta^{\mathrm{in}}(d\gamma) \\ &\geq \sum_{n=1}^{N_k} \int_{A^{\mathrm{in}}_{k,n}} \frac{1}{\sigma(\phi_\gamma^{\ell_j}(t_\gamma^-))} \bigg\{\int_{S_1\cap\{t=t_\gamma^-\}} \bigg[\int \mathbf{I}(t_n^-,t_n^+,\gamma',\ell_i)\, \eta^{\mathrm{in}}_{z'}(d\gamma')\bigg] \phi_\gamma^{\ell_j}(z')\rho(z')\,\mathcal{H}^d(dz')\bigg\}\,\eta^{\mathrm{in}}(d\gamma),\end{aligned}$$

because $\mathbf{I}(\gamma,\gamma',\ell_i) \supset \mathbf{I}(t_n^-,t_n^+,\gamma',\ell_i)$ when $\gamma \in A^{\mathrm{in}}_{k,n}$.

Using Fatou's Lemma, we conclude that

$$\begin{aligned}\varpi &\geq \liminf_{j\to+\infty} \sum_{k=1}^{N_k} \int_{A^{\mathrm{in}}_{k,n}} \frac{1}{\sigma(\phi_\gamma^{\ell_j}(t_\gamma^-))} \bigg\{\int_{S_1\cap\{t=t_\gamma^-\}} \bigg[\int \mathbf{I}(t_n^-,t_n^+,\gamma',\ell_i)\, \eta^{\mathrm{in}}_{z'}(d\gamma')\bigg] \phi_\gamma^{\ell_j}(z')\rho(z')\,\mathcal{H}^d(dz')\bigg\}\,\eta^{\mathrm{in}}(d\gamma) \\ &\geq \sum_k \int_{A^{\mathrm{in}}_{k,n}} \liminf_{j\to+\infty} \frac{1}{\sigma(\phi_\gamma^{\ell_j}(t_\gamma^-))} \bigg\{\int_{S_1\cap\{t=t_\gamma^-\}} \bigg[\int \mathbf{I}(t_n^-,t_n^+,\gamma',\ell_i)\, \eta^{\mathrm{in}}_{z'}(d\gamma')\bigg] \phi_\gamma^{\ell_j}(z')\rho(z')\,\mathcal{H}^d(dz')\bigg\}\,\eta^{\mathrm{in}}(d\gamma) \\ &\geq \sum_k \int_{A^{\mathrm{in}}_{k,n}} \bigg[\int \mathbf{I}(t_n^-,t_n^+,\gamma',\ell_i)\, \eta^{\mathrm{in}}_{\gamma(t_\gamma^-)}(d\gamma')\bigg]\,\eta^{\mathrm{in}}(d\gamma).\end{aligned} \tag{7.10}$$

By taking t_n^+ increasing and t_n^+ decreasing for η^{in}-a.e. γ, when $k \to \infty$ we have for every γ'

$$\sum_n \mathbf{I}(t_n^-,t_n^+,\gamma',\ell_i)\mathbb{1}_{A^{\mathrm{in}}_{k,n}} \nearrow \mathbf{I}(\gamma,\gamma',\ell_i),$$

on a η-conegligible set, so that by passing to the limit in n we conclude by monotonicity that

$$\varpi \geq \int_\Gamma \bigg[\int \mathbf{I}(\gamma,\gamma',\ell_i)\, \eta^{\mathrm{in}}_{\gamma(t_\gamma^-)}(d\gamma')\bigg]\,\eta^{\mathrm{in}}(d\gamma). \tag{7.11}$$

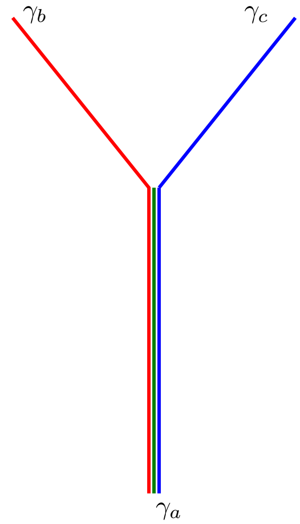


Figure 8. The discrete case described in Example 7.4: $\eta = \eta^{\mathrm{in}} = a\delta_{\gamma_a} + b\delta_{\gamma_b} + c\delta_{\gamma_c}$, where $a, b, c > 0$ are positive real numbers. The red and blue curves (resp. γ_b, γ_c are distinct but they have non trivial intersection, which coincides with γ_a, the green curve. It is clear that $\eta^{\mathrm{in}} \otimes \eta^{\mathrm{in}}(\mathit{\Gamma}^2 \setminus W) = bc + cb = 2bc$. On the other hand, if e.g. $b < c$, we can construct a plan which moves $b\delta_{\gamma_b}$ to $b\delta_{\gamma_c}$ and $b\delta_{\gamma_c}$ to $b\delta_{\gamma_b}$ (leaving the remaining $(c-b)\delta_{\gamma_c}$ fixed). For such a plan it holds $\pi(\mathit{\Gamma}^2 \setminus W) = b + b = 2b = 2\min\{b, c\}$.

Observe now that when $i \to +\infty$

$$\big\{(\gamma', \gamma'') : \phi_\gamma^{\ell_i}(\gamma''(\max\{t^-_{\gamma''}, t^-_\gamma\})) < a\big\} \nearrow \{(\gamma', \gamma'') : t^-_{\gamma''} \leq t^+_\gamma, \gamma''(\max\{t^-_{\gamma''}, t^-_\gamma\})) \notin \mathrm{Graph}\,\gamma\}) \\ = \{(\gamma', \gamma'') : t^-_{\gamma''} \leq t^+_\gamma, \gamma''(t^-_{\gamma''}) \notin \mathrm{Graph}\,\gamma\})$$

because for η^{in}-a.e. γ we have $\gamma(t^-_\gamma) \in S_1$. By Monotone Convergence Theorem, we then have

$$\begin{aligned} \varpi &\geq \int \bigg\{ \int \Big[\pi_{\gamma'}\big(\{(\gamma', \gamma'') : t^-_{\gamma''} \leq t^+_\gamma, \gamma''(t^-_{\gamma''}) \notin \mathrm{Graph}\,\gamma\} \cap \mathit{\Gamma}^2 \setminus W_1(\gamma)\big)\Big]\, \eta^{\mathrm{in}}_{\gamma(t^-_\gamma)}(d\gamma') \bigg\}\, \eta^{\mathrm{in}}(d\gamma) \\ &\geq \int \bigg\{ \int \Big[\pi_{\gamma'}\big(\{(\gamma', \gamma'') : \gamma''(t^-_{\gamma''}) \notin \mathrm{Graph}\,\gamma\} \cap \mathit{\Gamma}^2 \setminus W_1(\gamma)\big)\Big]\, \eta^{\mathrm{in}}_z \otimes \eta^{\mathrm{in}}_z(d\gamma d\gamma') \bigg\} \rho(z)\, \mathcal{H}^d(dz), \end{aligned} \tag{7.12}$$

which is what we wanted to prove taking into account the definition of $\mathit{\Gamma}^2 \setminus W_1(\gamma)$. □

Example 7.4. In general Proposition 7.2 is sharp and it holds

$$\eta^{\mathrm{in}} \otimes \eta^{\mathrm{in}}(\mathit{\Gamma}^2 \setminus W) < \pi(\mathit{\Gamma}^2 \setminus W),$$

so that we cannot expect a control on the quantity $\pi(\mathit{\Gamma}^2 \setminus W)$. For example, consider three curves γ_a, γ_b and γ_c starting at the same time ($t = 0$) such that

$$\gamma_a = \gamma_b \cap \gamma_c, \qquad \gamma_b \neq \gamma_c,$$

with weight a, b, c. See Figure 8.

Then one has

$$\eta^{\mathrm{in}} \otimes \eta^{\mathrm{in}}(\mathit{\Gamma}^2 \setminus W) = 2bc,$$

while by duality

$$\max \pi(\mathit{\Gamma}^2 \setminus W) = 2\min\{b, c\}.$$

Remark 7.5. By inspection, one can observe that to deduce Propositions 7.2 and 7.3 one can relax Point 2 to the following:

(2') for η^{in}-a.e. γ there are two sequences of Lipschitz functions $\phi^{\ell_i}_\gamma$, $\phi^{\ell_i{}'}_\gamma$ such that

(a) (7.3) is satisfied by $\phi^{\ell_i}_\gamma$ and

$$\phi^{\ell_i}_\gamma(\mathrm{Graph}\,\gamma) = 1, \qquad \lim_{i\to\infty} \mathrm{supp}\,\phi^{\ell_i}_\gamma = \mathrm{Graph}\,\gamma, \tag{7.13}$$

(b) (7.9) is satisfied by $\phi^{\ell_{i'}}_\gamma$ and

$$\phi^{\ell_{i'}}_\gamma(\mathrm{Graph}\,\gamma) = 1, \qquad \lim_{i'\to\infty} \mathrm{supp}\,\phi^{\ell_{i'}}_\gamma = \mathrm{Graph}\,\gamma, \tag{7.14}$$

(c) it holds

$$\lim_{\ell_i} \frac{\sigma((f\phi^{\ell_i}_\gamma)(t^-_\gamma))}{\sigma(\phi^{\ell_i}_\gamma(t^-_\gamma))} = f(\gamma(t^-_\gamma)), \qquad \lim_{\ell_{i'}} \frac{\sigma((f\phi^{\ell_{i'}}_\gamma)(t^-_\gamma))}{\sigma(\phi^{\ell_{i'}}_\gamma(t^-_\gamma))} = f(\gamma(t^-_\gamma)),$$

for all integrable functions f and η^{in}-a.e. γ, where $\sigma(\cdot)$ is defined in (7.2).

On can further require that (7.3), (7.9) hold up to a set of trajectories which vanishes when computing the limits (7.5), (7.10), and the same requirement for (7.13), (7.14).

Finally, in some cases it is easier to have replace ϕ^ℓ_γ with the characteristic function of an inner/outer proper set, replacing the integral of $\rho|(1,\mathbf{b})\cdot\mathbf{n}|$ with the inner/outer trace as follows.

Assumption 7.6 (Inner proper cylinders)**.** There are constants $\mathtt{M},\varpi > 0$ and a family of sets $\{Q^\ell_\gamma\}_{\ell>0,\gamma\in\Gamma}$ such that:

(1) for every $\gamma\in\Gamma, \ell\in\mathbb{R}^+$, the set $Q^\ell_\gamma\subset\mathbb{R}^{d+1}$ is $\rho(1,\mathbf{b})$-inner proper;
(2) for $t\in(t^-_\gamma,t^+_\gamma)$

$$\gamma(t) + B^d_{\ell/\mathtt{M}}(0) \subseteq Q_t \subseteq \gamma(t) + B^d_{\mathtt{M}\ell}(0);$$

(3) it holds

$$\int \biggl[\frac{1}{\sigma(\mathbb{1}_{Q^\ell_\gamma}(t^-_\gamma))}\int_{t\in(t^-_\gamma,t^+_\gamma)} \mathsf{Tr}\,\big(\rho(1,\mathbf{b}),Q\big)\mathcal{H}^d\llcorner_{\partial Q}\biggr]\eta^{\mathrm{in}}(d\gamma) \le \varpi, \tag{7.15}$$

where σ is given by (7.2).

The key observation is that being inner proper, up to an arbitrarily small quantity one can replace (7.15) with (7.1) because of Condition (3) of Definition 4.5. The two definitions are essentially equivalent because of Remark 4.3.

The assumption in the case of outer proper cylinders is analogous, and one can image also combinations of the two cases.

7.2. **Forward uniqueness.** We now turn our attention to the set of *crossing trajectories*, i.e. the trajectories which enter from S_1 and leave the domain Ω: set

$$\Gamma^{\mathrm{cr}} := \big\{\gamma : \gamma(t^-_\gamma)\in S_1, \gamma(t^+_\gamma)\in\partial\Omega\big\}$$

and define accordingly the measures

$$\eta^{\mathrm{cr}} := \eta\llcorner_{\Gamma^{\mathrm{cr}}}, \qquad \eta^{\mathrm{cr}}_z := \eta_z\llcorner_{\Gamma^{\mathrm{cr}}}.$$

Remark 7.7. Notice that $\|\eta^{\mathrm{cr}}_z\|$ may be less than 1, hence it is not the standard normalized disintegration of η^{cr} w.r.t. $\rho\mathcal{H}^d\llcorner_{S_1}$. By projection, the corresponding density $\rho^{\mathrm{cr}}\ge 0$, defined by

$$\rho^{\mathrm{cr}}(t,\cdot)\,\mathcal{L}^d = (e_t)_\#\eta^{\mathrm{cr}}$$

satisfies

$$\mathrm{div}\,\big(\rho^{\mathrm{cr}}(1,\mathbf{b})\big) = \rho^{\mathrm{cr}}\mathcal{H}^d\llcorner_{S_1} - \rho^{\mathrm{cr}}[(1,\mathbf{b})\cdot\mathbf{n}]^+\mathcal{H}^d\llcorner_{\partial\Omega}.$$

Furthermore, for $\mathcal{H}^d$-a.e. $z\in\partial\Omega$ it holds

$$\rho^{\mathrm{cr}}(z) = \|\eta^{\mathrm{cr}}_z\|\rho(z).$$

We start by observing that if $\gamma(t^+_\gamma)\in\partial\Omega$, then one can replace the requirement

$$\big(\gamma'\llcorner_{[t^-_\gamma,t^+_\gamma]},\gamma''\llcorner_{[t^-_\gamma,t^+_\gamma]}\big)\in\Gamma^2\setminus W_1$$

with

$$(\gamma',\gamma'')\in\Gamma^2\setminus W_1,$$

because in this case either $\gamma'\ne\gamma$ or $(\gamma,\gamma'')\in\Gamma^2\setminus W_1$: in particular (7.7) holds for all $(\gamma',\gamma'')\in\Gamma^2\setminus W_1$ for $\ell\ll 1$. By restricting the estimate in Proposition 7.3 to η^{cr}, we then deduce the following.

Corollary 7.8. *For any transport plan* $\pi \in \mathit{Adm}(\eta^{\mathrm{cr}}, \eta^{\mathrm{in}})$ *it holds*

$$\pi\big(\big\{(\gamma',\gamma''):\gamma''(t^-_{\gamma''}) \neq \gamma'(t^-_{\gamma'})\big\} \cap \mathit{\Gamma}^2 \setminus W_1\big) \leq \varpi. \tag{7.16}$$

Proof. Starting from (7.12), using the observation above and integrating, we obtain

$$\begin{aligned}
\varpi &\geq \int \bigg\{ \int \Big[\pi_{\gamma'}\big(\big\{(\gamma',\gamma''):\gamma''(t^-_{\gamma''}) \notin \operatorname{Graph}\gamma\big\} \cap \mathit{\Gamma}^2 \setminus W_1\big)\Big] \eta^{\mathrm{in}}_{\gamma(t^-_\gamma)}(d\gamma')\bigg\} \eta^{\mathrm{in}}(d\gamma) \\
&= \int \bigg\{ \int \Big[\pi_{\gamma'}\big(\big\{(\gamma',\gamma''):\gamma''(t^-_{\gamma''}) \neq \gamma(t^-_\gamma)\big\} \cap \mathit{\Gamma}^2 \setminus W_1\big)\Big] \eta^{\mathrm{in}}_{\gamma(t^-_\gamma)}(d\gamma')\bigg\} \eta^{\mathrm{in}}(d\gamma) \\
&= \int \bigg\{ \int \Big[\pi_{\gamma'}\big(\big\{(\gamma',\gamma''):\gamma''(t^-_{\gamma''}) \neq \gamma'(t^-_{\gamma'})\big\} \cap \mathit{\Gamma}^2 \setminus W_1\big)\Big] \eta^{\mathrm{in}}_z(d\gamma')\bigg\} \rho(z)\mathcal{H}^d(dz) \\
&= \pi\big(\big\{(\gamma',\gamma''):\gamma''(t^-_{\gamma''}) \neq \gamma'(t^-_{\gamma'})\big\} \cap \mathit{\Gamma}^2 \setminus W_1\big),
\end{aligned}$$

where we have used the observation that if γ, γ'' start on $\partial\Omega$ then the condition $\gamma''(t^-_{\gamma''}) \notin \operatorname{Graph}\gamma$ reduces to $\gamma''(t^-_{\gamma''}) \neq \gamma(t^-_\gamma) = \gamma'(t^-_{\gamma'}) = z$ by the domain of integration. □

Our goal now is to estimate in a quantitative way how much η^{cr} differs from a superposition of Dirac masses. This will be achieved using two main ingredients: on the one hand, we will use the estimates given by Proposition 7.2 and Proposition 7.3; on the other hand we will get rid of the divergence μ inside the domain Ω (which is the quantity which measures how many trajectories start or finish inside Ω) playing with constants.

Lemma 7.9. *It holds*

$$\int_{S_1} \big(\rho(z) - \rho^{\mathrm{cr}}(z)\big)\,\mathcal{H}^d(dz) \leq \mu^-(\Omega).$$

Proof. The balance of the divergence gives

$$\begin{aligned}
\int_{S_1} \big(\rho(z) - \rho^{\mathrm{cr}}(z)\big)\,\mathcal{H}^d(dz) &= \int_{S_1} \big(1 - \|\eta^{\mathrm{cr}}_z\|\big)\rho(z)\,\mathcal{H}^d(dz) \\
&= \int_{S_1} \eta_z(\Gamma \setminus \Gamma^{\mathrm{cr}})\rho(z)\,\mathcal{H}^d(dz) \leq \mu^-(\Omega),
\end{aligned}$$

because the curves which enter in S_1 but do not exit from Ω necessarily have the final point $\gamma(t^+_\gamma)$ inside Ω. □

Since clearly $\eta^{\mathrm{cr}} \leq \eta^{\mathrm{in}}$, by Proposition 7.2 we deduce the estimate

$$\int_{S_1} \eta^{\mathrm{cr}}_z \otimes \eta^{\mathrm{cr}}_z(\mathit{\Gamma}^2 \setminus W_2)\rho(z)\mathcal{H}^d(dz) \leq \varpi. \tag{7.17}$$

Observe now that, when we restrict to $\mathit{\Gamma}^{\mathrm{cr}}$, the following equality holds:

$$(\mathit{\Gamma}^{\mathrm{cr}})^2 \setminus W_2 = \big\{(\gamma,\gamma') \in (\mathit{\Gamma}^{\mathrm{cr}})^2 : \gamma \neq \gamma'\big\}.$$

Thus, we can rewrite (7.17) as

$$\int_{S_1} \eta^{\mathrm{cr}}_z \otimes \eta^{\mathrm{cr}}_z\big(\big\{(\gamma,\gamma') \in (\mathit{\Gamma}^{\mathrm{cr}})^2 : \gamma \neq \gamma'\big\}\big)\rho(z)\mathcal{H}^d(dz) \leq \varpi. \tag{7.18}$$

To proceed further, we need the following elementary lemma.

Lemma 7.10. *For any bounded, non negative measure* m *on a Polish space* Y *it holds*

$$\|\mathsf{m}\|\big(\|\mathsf{m}\| - \max_{y\in Y}\mathsf{m}(\{y\})\big) \leq \mathsf{m}\otimes\mathsf{m}(\{(y,y') :\, y \neq y'\}).$$

In particular, for probability measures

$$1 - \max_{y\in Y}\mathsf{m}(\{y\}) \leq \mathsf{m}\otimes\mathsf{m}(\{(y,y') :\, y \neq y'\}).$$

Proof. Decompose

$$\mathsf{m} = \mathsf{m}^{\mathrm{cont}} + \sum_n c_n \delta_{y_n},$$

so that

$$\mathsf{m} \otimes \mathsf{m}(\{(y,y') : y \neq y'\}) = \|\mathsf{m}\|^2 - \sum_n c_n^2.$$

Assume that

$$n \mapsto c_n$$

is decreasing, and estimate

$$\sum_n c_n^2 \leq c_1 \sum_n c_n \leq c_1 \|\mathsf{m}\|.$$

Hence

$$\mathsf{m} \otimes \mathsf{m}(\{(y,y') : y \neq y'\}) \geq \|\mathsf{m}\|(\|\mathsf{m}\| - c_1),$$

with

$$c_1 = \max_n c_n$$

which is the claim. □

Combining Proposition 7.2 (which gives (7.18)) with Lemma 7.10, we deduce the following proposition.

Proposition 7.11. *For any real constant $C > 1$, we have the estimate*

$$\int_{S_1} \big(\|\eta_z^{\mathrm{cr}}\| - \max_{\gamma \in \Gamma} \eta_z^{\mathrm{cr}}(\{\gamma\})\big)\rho(z)\, \mathcal{H}^d(dz) < C\varpi + \frac{\mu^-(\Omega)}{C-1}.$$

Proof. Write for $C > 1$

$$\begin{aligned}
&\int_{S_1} \bigg\{ \frac{\eta_z^{\mathrm{cr}}}{\|\eta_z^{\mathrm{cr}}\|} \otimes \frac{\eta_z^{\mathrm{cr}}}{\|\eta_z^{\mathrm{cr}}\|} \big(\{(\gamma,\gamma') : \gamma \neq \gamma'\}\big) \bigg\} \rho^{\mathrm{cr}}(z)\, \mathcal{H}^d(dz) \\
&\quad = \bigg[\int_{\rho^{\mathrm{cr}} \geq \rho/C} + \int_{\rho^{\mathrm{cr}} < \rho/C} \bigg] \bigg\{ \frac{\eta_z^{\mathrm{cr}}}{\|\eta_z^{\mathrm{cr}}\|} \otimes \eta_z^{\mathrm{cr}} \big(\{(\gamma,\gamma') : \gamma \neq \gamma'\}\big) \bigg\} \rho(z)\, \mathcal{H}^d \llcorner_{S_1}(dz) \\
&\quad \leq C \int_{\rho^{\mathrm{cr}} \geq \rho/C} \Big\{ \eta_z^{\mathrm{cr}} \otimes \eta_z^{\mathrm{cr}} \big((\Gamma^{\mathrm{cr}})^2 \setminus W_2\big) \Big\} \rho(z)\, \mathcal{H}^d \llcorner_{S_1}(dz) + \int_{\rho^{\mathrm{cr}} < \rho/C} \rho^{\mathrm{cr}}(z) \mathcal{H}^d \llcorner_{S_1}(dz) \\
&\quad < C\varpi + \frac{1}{C} \frac{\mu^-(\Omega)}{1 - 1/C},
\end{aligned}$$

where in the last passage we have used Lemma 7.9. Now the conclusion follows directly applying Lemma 7.10. □

From Proposition 7.11, we deduce that, up to a set of trajectories whose η-measure is controlled, the measure η^{cr} is essentially a superposition of Dirac deltas. More precisely, we can find a family of crossing trajectories $\varXi \subset \Gamma^{\mathrm{cr}}$ such that

$$\eta^{\mathrm{cr}}(\Gamma^{\mathrm{cr}} \setminus \varXi) < C\varpi + \frac{\mu^-(\Omega)}{C-1}$$

and

$$(\eta^{\varXi})_z = \eta_z^{\mathrm{cr}} \llcorner_{\varXi} := m_z \delta_{\gamma_z}, \qquad \gamma_z \in \Gamma^{\mathrm{cr}}. \tag{7.19}$$

This additional piece of information can be combined together with Proposition 7.2 in the following way.

Consider an admissible plan $\tilde{\pi} \in \mathsf{Adm}(\eta^{\Xi}, \eta^{\mathrm{in}})$. We have the following lemma.

Lemma 7.12. *Let*

$$\mathcal{S} := \big\{(\gamma,\gamma') : \gamma(t_\gamma^-) = \gamma'(t_{\gamma'}^-)\big\} \subset \Gamma^2,$$

i.e. the set of curves which start from the same point. Then

$$\tilde{\pi} \llcorner_{\mathcal{S}} \big(\Gamma^2 \setminus W_2\big) \leq \varpi. \tag{7.20}$$

Proof. By Disintegration Theorem (applied w.r.t. the map $\mathcal{S} \ni (\gamma, \gamma') \mapsto \gamma(t_\gamma^-)$), we have

$$\tilde{\pi}\llcorner_{\mathcal{S}} = \int_{S_1} \big(\tilde{\pi}\llcorner_{\mathcal{S}}\big)_z \rho(z)\, \mathcal{H}^d(dz),$$

where $(\tilde{\pi}\llcorner_{\mathcal{S}})_z \in \mathsf{Adm}(\eta_z^{\Xi}, \eta_z^{\rm in})$ for $\mathcal{H}^d$-a.e. $z \in S_1$. Being η_z^{Ξ} the Dirac delta $m_z \delta_{\gamma_z}$ in view of (7.19), it follows that every transference plan in $\tilde{\pi}_z \in \mathsf{Adm}(\eta_z^{\Xi}, \eta_z^{\rm in})$ satisfies

$$\tilde{\pi}_z \leq \eta_z^{\Xi} \otimes \eta_z^{\rm in} \leq \eta_z^{\rm in} \otimes \eta_z^{\rm in},$$

so that Proposition 7.2 directly implies the statement. $\square$

By summing up the results in Lemma 7.12 and Corollary 7.8 we deduce the following corollary.

Corollary 7.13. *For any admissible transport plan $\pi \in \mathsf{Adm}(\eta^{\rm cr}, \eta^{\rm in})$, it holds*

$$\pi(\mathit{\Gamma}^2 \setminus W) < 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1}.$$

Proof. For any plan π we have

$$\begin{aligned} \pi(\mathit{\Gamma}^2 \setminus W) &= \pi\big((\mathit{\Xi} \times \mathit{\Gamma}) \setminus W\big) + \pi\big(((\mathit{\Gamma} \setminus \mathit{\Xi}) \times \mathit{\Gamma}) \setminus W\big) \\ &\leq \pi((\mathit{\Xi} \times \mathit{\Gamma}) \setminus W) + \eta^{\rm cr}(\mathit{\Gamma} \setminus \mathit{\Xi}) \\ \text{by (7.19) and Proposition 7.11} \quad &\leq \pi((\mathit{\Xi} \times \mathit{\Gamma}) \setminus W) + C\varpi + \frac{\mu^-(\Omega)}{C-1} \\ &\leq 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1}, \end{aligned}$$

where in the last line we have use the fact that $\pi\llcorner_{\mathit{\Xi}\times\mathit{\Gamma}} \in \mathsf{Adm}(\eta^{\Xi}, \eta^{\rm in})$ so that (7.20) and (7.16) give the estimate. $\square$

Notice that we can rephrase Corollary 7.13 by saying that

$$\sup_{\pi \in \mathsf{Adm}(\eta^{\rm cr}, \eta^{\rm in})} \pi(\mathit{\Gamma}^2 \setminus W) \leq 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1}. \tag{7.21}$$

for all $C > 1$.

Invoking the deep duality results of [Kel84] recalled in Section 3.2, we can prove the following

Theorem 7.14. *There exist Borel sets $N_1 \subset \mathit{\Gamma}^{\rm cr}, N_2 \subset \mathit{\Gamma}^{\rm in}$ such that*

$$\eta^{\rm cr}(N_1) + \eta^{\rm in}(N_2) \leq 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1},$$

and for every $(\gamma, \gamma') \in (\mathit{\Gamma}^{\rm cr} \setminus N_1) \times (\mathit{\Gamma}^{\rm in} \setminus N_2)$ either $\operatorname{Graph}\gamma' \subset \operatorname{Graph}\gamma$ *or* $\operatorname{Graph}\gamma \cap \operatorname{Graph}\gamma' = \emptyset$.

Equivalently we can say that

$$(\mathit{\Gamma}^{\rm cr} \setminus N_1) \times (\mathit{\Gamma}^{\rm in} \setminus N_2) \subset W.$$

Proof. Taking into account Theorem 3.2 and Proposition 3.3, we have that there exist Borel sets N_1, N_2 such that

$$\mathbb{1}_{N_1} + \mathbb{1}_{N_2} \geq \mathbb{1}_{(\mathit{\Gamma}^{\rm cr} \times \mathit{\Gamma}) \setminus W}$$

and

$$\eta^{\rm cr}(N_1) + \eta^{\rm in}(N_2) = \sup_{\pi \in \mathsf{Adm}(\eta^{\rm cr}, \eta^{\rm in})} \pi(\mathit{\Gamma}^2 \setminus W) \overset{(7.21)}{\leq} 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1},$$

which is exactly the claim. $\square$

Recall now that, so far, we have been working with $\Omega = \Omega^\varepsilon$, being Ω a proper set and $\Omega^\varepsilon \supset \Omega$ the perturbed set constructed in Proposition 4.15. In some sense, we now want to pass to the limit the above estimates as $\varepsilon \to 0$.

Let $\Omega \subset \mathbb{R}^{d+1}$ be a proper set and η be a Lagrangian representation of $\rho(1, \mathbf{b})\, \mathcal{L}^{d+1}$. Set

$$\mathit{\Gamma}^{\rm cr}(\Omega) := \big\{\gamma \in \mathit{\Gamma} : \gamma(t_\gamma^\pm) \in \partial\Omega\big\}, \qquad \mathit{\Gamma}^{\rm in}(\Omega) := \big\{\gamma \in \mathit{\Gamma} : \gamma(t_\gamma^-) \in \partial\Omega\big\}.$$

Assume that Theorem 7.14 holds for a family of perturbations Ω^{ϵ_n} with constant ϖ.

Theorem 7.15. *There exist $N_1 \subset \Gamma^{\rm cr}(\Omega), N_2 \subset \Gamma^{\rm in}(\Omega)$ such that*

$$(\mathtt{R}_\Omega)_\sharp \eta^{\rm cr}(N_1) + (\mathtt{R}_\Omega)_\sharp \eta^{\rm in}(N_2) \leq \inf_{C>1} \left\{ 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1} \right\}$$

and for every $(\gamma,\gamma') \in (\Gamma^{\rm cr} \setminus N_1) \times (\Gamma^{\rm in} \setminus N_2)$ either

$$\mathrm{Graph}\, \gamma' \llcorner_{\mathrm{clos}\,\Omega} \subset \mathrm{Graph}\, \gamma \llcorner_{\mathrm{clos}\,\Omega} \quad or \quad \mathrm{Graph}\, \gamma \llcorner_{\mathrm{clos}\,\Omega} \cap \mathrm{Graph}\, \gamma' \llcorner_{\mathrm{clos}\,\Omega} = \emptyset.$$

Proof. From Theorem 7.14 applied to every Ω^{ε_n}, we obtain two sets $N_1^{\varepsilon_n}$ and $N_2^{\varepsilon_n}$ such that

$$(\mathtt{R}_{\Omega^{\epsilon_n}})_\sharp \eta^{\rm cr}(N_1^{\varepsilon_n}) + (\mathtt{R}_{\Omega^{\epsilon_n}})_\sharp \eta^{\rm in}(N_2^{\varepsilon_n}) \leq 2\varpi + C\varpi + \frac{\mu^-(\Omega^{\epsilon_n})}{C-1},$$

and for every $(\gamma,\gamma') \in (\Gamma^{\rm cr}(\Omega^{\varepsilon_n}) \setminus N_1^{\varepsilon_n}) \times (\Gamma^{\rm in}(\Omega^{\varepsilon_n}) \setminus N_2^{\varepsilon_n})$ either

$$\mathrm{Graph}\, \gamma' \llcorner_{\mathrm{clos}\,\Omega^{\varepsilon_n}} \subset \mathrm{Graph}\, \gamma \llcorner_{\mathrm{clos}\,\Omega^{\varepsilon_n}} \quad \text{or} \quad \mathrm{Graph}\, \gamma' \llcorner_{\mathrm{clos}\,\Omega^{\varepsilon_n}} \cap \mathrm{Graph}\, \gamma \llcorner_{\mathrm{clos}\,\Omega^{\varepsilon_n}} = \emptyset.$$

Now $\mathtt{R}_\Omega(\Gamma^{\rm cr}(\Omega^{\varepsilon_n})) \subset \Gamma^{\rm cr}(\Omega)$ and

$$\big|(\mathtt{R}_\Omega)_\sharp \eta(\Gamma^{\rm cr}(\Omega)) - (\mathtt{R}_{\Omega^{\varepsilon_n}})_\sharp \eta(\Gamma^{\rm cr}(\Omega^{\varepsilon_n}))\big| < \mathcal{O}(\varepsilon_n)$$

from Theorem 4.16 and the estimates therein. In the same way, $\mathtt{R}_\Omega(\Gamma^{\rm in}(\Omega^{\varepsilon_n})) \subset \Gamma^{\rm in}(\Omega)$ and

$$\big|(\mathtt{R}_\Omega)_\sharp \eta(\Gamma^{\rm in}(\Omega)) - (\mathtt{R}_{\Omega^{\varepsilon_n}})_\sharp \eta(\Gamma^{\rm in}(\Omega^{\varepsilon_n}))\big| < \mathcal{O}(\varepsilon_n).$$

If we now consider the sets

$$\tilde N_1^{\varepsilon_n} := \mathtt{R}_\Omega(N_1^{\varepsilon_n}) \cup \big(\Gamma^{\rm cr}(\Omega) \setminus \mathtt{R}_\Omega\big(\Gamma^{\rm cr}(\Omega^{\varepsilon_n})\big)\big) \quad \text{and} \quad \tilde N_2^{\varepsilon_n} := \mathtt{R}_\Omega(N_2^{\varepsilon_n}) \cup \big(\Gamma^{\rm in}(\Omega) \setminus \mathtt{R}_\Omega\big(\Gamma^{\rm in}(\Omega^{\varepsilon_n})\big)\big),$$

we have by Corollary 6.11 as $\varepsilon_n \to 0$ that

$$(\mathtt{R}_\Omega)_\sharp \eta^{\rm cr}(\tilde N_1^{\varepsilon_n}) + (\mathtt{R}_\Omega)_\sharp \eta^{\rm in}(\tilde N_2^{\varepsilon_n}) \leq 2\varpi + C\varpi + \frac{\mu^-(\Omega^{\epsilon_n})}{C-1} + o(1) = 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1} + o(1),$$

and for every $(\gamma,\gamma') \in (\Gamma^{\rm cr}(\Omega) \setminus \tilde N_1^{\varepsilon_n}) \times (\Gamma^{\rm in}(\Omega) \setminus \tilde N_2^{\varepsilon_n})$ either

$$\mathrm{Graph}\, \gamma' \llcorner_{\mathrm{clos}\,\Omega} \subset \mathrm{Graph}\, \gamma \llcorner_{\mathrm{clos}\,\Omega} \quad \text{or} \quad \mathrm{Graph}\, \gamma' \llcorner_{\mathrm{clos}\,\Omega} \cap \mathrm{Graph}\, \gamma \llcorner_{\mathrm{clos}\,\Omega} = \emptyset.$$

In particular, it follows that

$$\inf \left\{ (\mathtt{R}_\Omega)_\sharp \eta^{\rm cr}(N_1) + (\mathtt{R}_\Omega)_\sharp \eta^{\rm in}(N_2) : (\Gamma^{\rm cr} \setminus N_1) \times (\Gamma^{\rm in} \setminus N_2) \subset W \right\} \leq 2\varpi + C\varpi + \frac{\mu^-(\Omega)}{C-1},$$

and we apply again Proposition 3.3 in order to find two actual minimizers. □

8. Untangling functional and untangled Lagrangian representations

This section is divided into two parts. In the first part, following the analysis of Theorem 7.15, we define two functionals on the family of proper sets which measure how much the trajectories used by a Lagrangian representation η cross each other. The main result is that these functionals are subadditive, so that it seems natural to compare them with a measure ϖ^τ. This is the main result of the second part, which shows that if one can bound the untangling functional in sufficiently many sets by a given measure, then we can have an estimate on how many trajectories one has to remove in order to obtain an untangled set of trajectories, i.e. trajectories which do not cross each other.

8.1. Subadditivity of untangling functional.

For $\Omega \subset \mathbb{R}^{d+1}$ proper set we give the following definition.

Definition 8.1. The *untangling functional for $\eta^{\rm in}$* is defined as

$$\mathfrak{f}^{\rm in}(\Omega) := \inf \Big\{ (\mathtt{R}_\Omega)_\sharp \eta^{\rm cr}(N_1) + (\mathtt{R}_\Omega)_\sharp \eta^{\rm in}(N_2) : \ (\Gamma \setminus N_1) \times (\Gamma \setminus N_2) \subset W \Big\}. \tag{8.1}$$

Setting

$$(\mathtt{R}_\Omega)_\sharp \eta^{\rm out} := \int_{\partial\Omega} \eta_z \rho(z)[(1,\mathbf{b}(z)) \cdot \mathbf{n}(z)]^+ \mathcal{H}^d(dz),$$

we can define analogously the untangling functional for $\eta^{\rm out}$.

Definition 8.2. The *untangling functional for $\eta^{\rm out}$* is defined as

$$\mathfrak{f}^{\rm out}(\Omega) := \inf \Big\{ (\mathtt{R}_\Omega)_\sharp \eta^{\rm cr}(N_1) + (\mathtt{R}_\Omega)_\sharp \eta^{\rm out}(N_2) : \ (\Gamma \setminus N_1) \times (\Gamma \setminus N_2) \subset W \Big\}. \tag{8.2}$$

As noticed before, the condition $(\gamma,\gamma') \in (\Gamma^{\mathrm{cr}}(\Omega) \times \Gamma^{\mathrm{in}}(\Omega)) \cap W$ is equivalent to say that

$$\text{either} \quad \operatorname{Graph} \gamma' \llcorner_{\operatorname{clos}\Omega} \subset \operatorname{Graph} \gamma \llcorner_{\operatorname{clos}\Omega} \quad \text{or} \quad \operatorname{Graph} \gamma \llcorner_{\operatorname{clos}\Omega} \cap \operatorname{Graph} \gamma' \llcorner_{\operatorname{clos}\Omega} = \emptyset,$$

and similarly for $(\gamma,\gamma') \in (\Gamma^{\mathrm{cr}} \times \Gamma^{\mathrm{out}}) \cap W$. Recalling now Theorem 7.15 we can infer that the infima in (8.1) and (8.2) are actually minima.

We now show the following remarkable property of the untangling functionals:

Proposition 8.3. *The functionals $\mathfrak{f}^{\mathrm{in}}$ and $\mathfrak{f}^{\mathrm{out}}$ are subadditive on the class of proper sets. More precisely, if $\mathrm{U},\mathrm{V} \subset \mathbb{R}^{d+1}$ are proper sets whose union $\Omega := \mathrm{U} \cup \mathrm{V}$ is proper, then*

$$\mathfrak{f}^{\mathrm{in}}(\Omega) \le \mathfrak{f}^{\mathrm{in}}(\mathrm{U}) + \mathfrak{f}^{\mathrm{in}}(\mathrm{V}), \qquad \mathfrak{f}^{\mathrm{out}}(\Omega) \le \mathfrak{f}^{\mathrm{out}}(\mathrm{U}) + \mathfrak{f}^{\mathrm{out}}(\mathrm{V}).$$

Proof. We prove the assertion only for the functional $\mathfrak{f}^{\mathrm{in}}$, being the other case completely similar. By definition, there exist sets $N_1(\mathrm{U}) \subset \Gamma^{\mathrm{cr}}(\mathrm{U})$ and $N_2(\mathrm{U}) \subset \Gamma^{\mathrm{in}}(\mathrm{U})$ such that

$$\mathfrak{f}^{\mathrm{in}}(\mathrm{U}) = (\mathtt{R}_{\mathrm{U}})_\sharp \eta^{\mathrm{cr}}(N_1(\mathrm{U})) + (\mathtt{R}_{\mathrm{U}})_\sharp \eta^{\mathrm{in}}(N_2(\mathrm{U}))$$

and

$$\big(\Gamma^{\mathrm{cr}}(\mathrm{U}) \setminus N_1(\mathrm{U})\big) \times \big(\Gamma^{\mathrm{in}}(\mathrm{U}) \setminus N_2(\mathrm{U})\big) \subset W.$$

Let $N_1(\mathrm{V}), N_2(\mathrm{V})$ be a corresponding couple of sets for V. Set

$$N_1 := \big\{\gamma \in \Gamma^{\mathrm{cr}}(\Omega) : \exists i \,\big(\mathtt{R}^i_{\mathrm{U}}\gamma \in N_1(\mathrm{U})\big)\big\} \cup \big\{\gamma \in \Gamma^{\mathrm{cr}}(\Omega) : \exists i \,\big(\mathtt{R}^i_{\mathrm{V}}\gamma \in N_1(\mathrm{V})\big)\big\}$$

and

$$N_2 := \big\{\gamma \in \Gamma^{\mathrm{in}}(\Omega) : \exists i \,\big(\mathtt{R}^i_{\mathrm{U}}\gamma \in N_2(\mathrm{U})\big)\big\} \cup \big\{\gamma \in \Gamma^{\mathrm{in}}(\Omega) : \exists i \,\big(\mathtt{R}^i_{\mathrm{V}}\gamma \in N_2(\mathrm{V})\big)\big\}.$$

By Proposition 6.10

$$\begin{aligned}
\eta(N_1) + \eta(N_2) &\le \eta\big(\big\{\gamma \in \Gamma^{\mathrm{cr}}(\Omega) : \exists i \,\big(\mathtt{R}^i_{\mathrm{U}}(\gamma) \in N_1(\mathrm{U})\big)\big\}\big) + \eta\big(\big\{\gamma \in \Gamma^{\mathrm{cr}}(\Omega) : \exists i \,\big(\mathtt{R}^i_{\mathrm{U}}(\gamma) \in N_2(\mathrm{U})\big)\big\}\big) \\
&\quad + \eta\big(\big\{\gamma \in \Gamma^{\mathrm{cr}}(\Omega) : \exists i \,\big(\mathtt{R}^i_{\mathrm{V}}(\gamma) \in N_1(\mathrm{V})\big)\big\}\big) + \eta\big(\big\{\gamma \in \Gamma^{\mathrm{cr}}(\Omega) : \exists i \,\big(\mathtt{R}^i_{\mathrm{V}}(\gamma) \in N_2(\mathrm{V})\big)\big\}\big) \\
&\le (\mathtt{R}_{\mathrm{U}})_\sharp \eta(N_1(\mathrm{U})) + (\mathtt{R}_{\mathrm{U}})_\sharp \eta(N_2(\mathrm{U})) + (\mathtt{R}_{\mathrm{V}})_\sharp \eta(N_1(\mathrm{V})) + (\mathtt{R}_{\mathrm{V}})_\sharp \eta(N_2(\mathrm{V})) \\
&= \mathfrak{f}^{\mathrm{in}}(\mathrm{U}) + \mathfrak{f}^{\mathrm{in}}(\mathrm{V}).
\end{aligned}$$

It remains to show $(\Gamma^{\mathrm{cr}}(\Omega) \setminus N_1) \times (\Gamma^{\mathrm{in}}(\Omega) \setminus N_2) \subset W$: this follows from the observation

$$\mathtt{R}_{\mathrm{U}}(\Gamma^{\mathrm{cr}}(\Omega)) \subset \Gamma^{\mathrm{cr}}(\mathrm{U}),$$

and

$$\mathtt{R}_{\mathrm{U}}(\Gamma^{\mathrm{in}}(\Omega)) \subset \Gamma^{\mathrm{in}}(\mathrm{U})$$

and the same for V. Hence, if $\operatorname{Graph} \gamma \llcorner_{\operatorname{clos}\Omega} \cap \operatorname{Graph} \gamma' \llcorner_{\operatorname{clos}\Omega} \neq \emptyset$ then they must coincide either in $\operatorname{clos}\mathrm{U}$ or $\operatorname{clos}\mathrm{V}$ and, by elementary arguments, in $\operatorname{clos}\mathrm{U} \cup \operatorname{clos}\mathrm{V} = \operatorname{clos}\Omega$. □

We conclude this paragraph with the following lemma, which shows that $\mathfrak{f}^{\mathrm{in}}$ and $\mathfrak{f}^{\mathrm{out}}$ are related.

Lemma 8.4. *It holds*

$$\mathfrak{f}^{\mathrm{in}}(\Omega) - \mu^-(\Omega) \le \mathfrak{f}^{\mathrm{out}}(\Omega) \le \mathfrak{f}^{\mathrm{in}}(\Omega) + \mu^+(\Omega)$$

where we recall that μ^+, μ^- are the positive/negative part of the measure $\mu = \operatorname{div}(\rho(1,\mathbf{b}))$.

Proof. We prove only $\mathfrak{f}^{\mathrm{out}}(\Omega) \le \mathfrak{f}^{\mathrm{in}}(\Omega) + \mu^+(\Omega)$, the other case being analogous. Let η be a Lagrangian representation of $\rho(1,\mathbf{b})\mathcal{L}^{d+1}\llcorner_\Omega$, and N_1, N_2 a minimal couple for $\mathfrak{f}^{\mathrm{in}}$. Since

$$\eta^{\mathrm{cr}}(N_2) \le \eta^{\mathrm{in}}(N_2),$$

then it follows that $(\Gamma^{\mathrm{cr}}(\Omega) \setminus (N_1 \cup N_2))^2 \subset W$. As already observed in Lemma 7.9,

$$\|\eta^{\mathrm{out}} - \eta^{\mathrm{cr}}\| \le \mu^+(\Omega),$$

so that the conclusion follows by considering the couple $N_1' = N_1 \cup N_2$ and $N_2' = \{\gamma : \gamma(t^-_\gamma) \in \Omega\}$. □

8.2. **Untangled Lagrangian representations.** Assume the following:

Assumption 8.5. Let $\tau > 0$ and $C > 1$ be such that

(1) there exist $K^{\tau,\pm}$ compact sets satisfying

$$\mu^\pm(K^{\tau,\mp}) = 0, \qquad \mu^\pm\big(\mathbb{R}^{d+1} \setminus K^{\tau,\pm}\big) < \tau;$$

(2) there exists a positive measure ϖ^τ such that

(a) for all $(t,x) \in K^{\tau,-}$ there exists a family of proper balls $\{B^{d+1}_r(t,x)\}_r$ with 0 as Lebesgue density point and such that it holds

$$\mathfrak{f}^{\mathrm{in}}(B^{d+1}_r(t,x)) \leq (C+2)\varpi^\tau(B^{d+1}_r(t,x)) + \frac{\mu^-(B^{d+1}_r(t,x))}{C-1},$$

(b) for all $(t,x) \in K^{\tau,+}$ there exists a family of proper balls $\{B^{d+1}_r(t,x)\}_r$ with 0 as Lebesgue density point and such that it holds

$$\mathfrak{f}^{\mathrm{out}}(B^{d+1}_r(t,x)) \leq (C+2)\varpi^\tau(B^{d+1}_r(t,x)) + \frac{\mu^+(B^{d+1}_r(t,x))}{C-1},$$

(c) for all $(t,x) \in \mathbb{R}^{d+1} \setminus (K^{\tau,-} \cup K^{\tau,+})$ there exists a family of proper balls $\{B^{d+1}_r(t,x)\}_r$ with 0 as Lebesgue density point and such that it holds

$$\min\big\{\mathfrak{f}^{\mathrm{in}}(B^{d+1}_r(t,x)), \mathfrak{f}^{\mathrm{out}}(B^{d+1}_r(t,x))\big\} \leq (C+2)\varpi^\tau(B^{d+1}_r(t,x)) + \frac{|\mu|(B^{d+1}_r(t,x))}{C-1}.$$

By the choice of the sets $K^{\tau,\pm}$ we can have in a sufficiently small ball the following estimate.

Proposition 8.6. *For every $(t,x) \in \mathbb{R}^{d+1}$ there exists $r_{t,x}$ such that for the families of balls $\{B^{d+1}_r(t,x)\}_r$ as above and for $r < r_{t,x}$ it holds*

$$\begin{aligned}\mathfrak{f}^{\mathrm{in}}(B^{d+1}_r(t,x)), \mathfrak{f}^{\mathrm{out}}(B^{d+1}_r(t,x)) \leq (C+2)\varpi^\tau(B^{d+1}_r(t,x)) &+ \frac{|\mu|(B^{d+1}_r(t,x))}{C-1} \\ &+ \frac{C}{C-1}|\mu|(B^{d+1}_r(t,x) \setminus K^{\tau,+} \cup K^{\tau,-}).\end{aligned} \tag{8.3}$$

Proof. It $(t,x) \in K^{\tau,-}$, then by Point (2a) of Assumption 8.5

$$\mathfrak{f}^{\mathrm{in}}(B^{d+1}_r(t,x)) \leq (C+2)\varpi^\tau(B^{d+1}_r(t,x)) + \frac{\mu^-(B^{d+1}_r(t,x))}{C-1},$$

and since $(t,x) \in K^{\tau,-}$, by Point (1) we can take $r \ll 1$ such that

$$\mu^+(B^{d+1}_r(t,x)) \leq \frac{\mu^-(B^{d+1}_r(t,x))}{C-1}.$$

One thus applies the Lemma 8.4 above. A completely similar computation holds for K^+.

For points in the open set $\mathbb{R}^{d+1} \setminus (K^{\tau,-} \cup K^{\tau,+})$ just take a ball $B^{d+1}_r(t,x) \subset \mathbb{R}^{d+1} \setminus (K^{\tau,-} \cup K^{\tau,+})$ and combine Point (2c) and Lemma 8.4. ☐

For future reference let us define the measure

$$\zeta^\tau_C := (C+2)\varpi^\tau + \frac{|\mu|}{C-1} + \frac{C}{C-1}|\mu|\llcorner_{\mathbb{R}^{d+1} \setminus K^{\tau,+} \cup K^{\tau,-}}.$$

A covering argument yields the following global estimate.

Corollary 8.7. *If $\Omega \subset \mathbb{R}^{d+1}$ is a proper set with compact closure, then*

$$\mathfrak{f}^{\mathrm{in}}(\Omega), \mathfrak{f}^{\mathrm{out}}(\Omega) \leq C_d \zeta^\tau_C(\operatorname{clos}\Omega), \tag{8.4}$$

where C_d is a dimensional constant.

Proof. Thanks to Proposition 8.6 and Vitali Theorem, for any $\varepsilon > 0$, we can cover the compact set $\operatorname{clos}\Omega$ with finitely many proper balls B_i such that the estimates (8.3) hold and

$$\sum_i \zeta^\tau_C(B_i) \leq C_d \zeta^\tau_C(\operatorname{clos}\Omega) + \varepsilon.$$

Thanks to the subadditivity (and the monotonicity) of $\mathfrak{f}^{\mathrm{in}}$ we can thus write

$$\mathfrak{f}^{\mathrm{in}}(\Omega) \le \mathfrak{f}^{\mathrm{in}}\Big(\bigcup_i B_i\Big) \le \sum_i \mathfrak{f}^{\mathrm{in}}(B_i) \le C_d \zeta_C^\tau(\operatorname{clos}\Omega) + \varepsilon.$$

Sending $\varepsilon \to 0$ we obtain (8.4). The same proof holds for the functional $\mathfrak{f}^{\mathrm{out}}$. $\square$

Let now $N \subset \Gamma$ be a set such that

$$(\Gamma \setminus N)^2 \subset \mathring{W},$$

where

$$\begin{aligned}\mathring{W} = &\Big\{(\gamma,\gamma') : \operatorname{Graph}\gamma\llcorner_{(t^-_\gamma, t^+_\gamma)} \cap \operatorname{Graph}\gamma'_{(t^-_{\gamma'}, t^+_{\gamma'})} = \emptyset\Big\} \\ &\cup \Big\{(\gamma,\gamma') : \operatorname{Graph}\gamma \cap \operatorname{Graph}\gamma' = \operatorname{Graph}\big(\gamma\llcorner_{[\max\{t^-_\gamma, t^-_{\gamma'}\}, \min\{t^+_\gamma, t^+_{\gamma'}\}]}\big)\Big\}.\end{aligned}$$

In the last part of this section we want estimate the measure $\eta(N)$ in terms of $\zeta_C^\tau(\mathbb{R}^{d+1}) = \|\zeta_C^\tau\|$. To this aim, define the compact sets (recall we consider solutions in a bounded domain)

$$\mathcal{K}^n := \big\{\gamma \in \Gamma : t^+_\gamma - t^-_\gamma \ge 2^{1-n}\big\}$$

and observe that, given $\varepsilon > 0$ there exists $n \gg 1$ such that

$$\eta(\Gamma \setminus \mathcal{K}^n) \le \epsilon.$$

If $(\gamma,\gamma') \in \Gamma^2 \setminus \mathring{W}$, then there exists $n \in \mathbb{N}$ such that $\gamma, \gamma' \in \mathcal{K}^n$ and

$$\operatorname{Graph}\gamma\llcorner_{[t^-_\gamma + 2^{-n}, t^+_\gamma - 2^{-n}]} \cap \operatorname{Graph}\gamma' \neq \emptyset, \tag{8.5a}$$

$$\sup\Big\{|\gamma(t) - \gamma'(t)|, t \in \big[\max\{t^-_\gamma + 2^{-n}, t^-_{\gamma'}\}, \min\{t^+_\gamma - 2^{-n}, t^+_{\gamma'}\}\big]\Big\} > 0, \tag{8.5b}$$

so that we can write

$$\Gamma^2 \setminus \mathring{W} = \bigcup_n Z^n$$

where

$$Z^n := \big\{(\gamma,\gamma') \in (\mathcal{K}^n)^2 : \text{(8.5) holds}\big\}.$$

Now consider a covering of the compact set

$$K^n := \bigcup_{\gamma \in \mathcal{K}^n} \operatorname{Graph}\gamma\llcorner_{[t^-_\gamma + 2^{-n}, t^+_\gamma - 2^{-n}]}$$

made up of finitely many proper balls $B_i := B^{d+1}_{r_i}(t_i, x_i)$ with radius less than 2^{-n}, for which Proposition 8.6 holds together with $\zeta_C^\tau(\partial B_i) = 0$, and define

$$O^n := \bigcup_i B_i.$$

We now have the following lemma, whose proof is elementary.

Lemma 8.8. *If $(\gamma,\gamma') \in Z^n$ then*

a) *if $\operatorname{Graph}\gamma \cap B_i \neq \emptyset$ then $\mathtt{R}_{B_i}\gamma \in \Gamma^{\mathrm{cr}}(B_i)$;*
b) *if $\operatorname{Graph}\gamma' \cap B_i \neq \emptyset$ then $\mathtt{R}_{B_i}\gamma \in \Gamma^{\mathrm{in}}(B_i) \cup \Gamma^{\mathrm{out}}(B_i)$;*
c) *there exists i such that $(\mathtt{R}_{B_i}\gamma, \mathtt{R}_{B_i}\gamma') \notin W$.*

Applying Corollary 8.7, we obtain $N^n_1 \subset \mathcal{K}^n$ and $N^n_2 \subset \mathcal{K}^n$ such that

$$\eta(N^n_1) + \eta(N^n_2) \le C_d \zeta_C^\tau(\operatorname{clos} O^n) = C_d \zeta_C^\tau(O^n)$$

and

$$\mathtt{R}_{\operatorname{clos} O^n}(\mathcal{K}^n \setminus N^n_1) \times \mathtt{R}_{\operatorname{clos} O^n}(\mathcal{K}^n \setminus N^n_2) \subset \Gamma^2 \setminus Z^n.$$

Now send $n \to +\infty$ with the same reasoning of Theorem 7.14 we finally obtain the following result.

Theorem 8.9. *There exists a set $N \subset \Gamma$ such that*

$$\eta(N) \leq C_d \zeta_\tau^C(\mathbb{R}^{d+1})$$

and

$$(\Gamma \setminus N)^2 \subset \mathring{W}.$$

The following definition seems now natural:

Definition 8.10. A Lagrangian representation η is called *untangled* if there exists a set $\Delta \subset \Gamma$ such that

a) $\Delta \times \Delta \subset \mathring{W}$ and
b) η is concentrated on Δ.

By inner regularity we can assume Δ to be σ-compact. We conclude by pointing out the following important point.

Corollary 8.11. *Suppose there exist sequences $\tau_i \searrow 0$ and $C_i \nearrow +\infty$ such that Assumption 8.5 holds for τ_i, C_i and moreover*

$$C_i \|\varpi^{\tau_i}\| \to 0.$$

Then η is untangled.

Proof. It is enough to observe that $\zeta_{C_i}^{\tau_i} \to 0$. □

Notice that the assumptions of the above corollary are satisfied if one assumes that in each point of the compact sets $K^{\tau,\pm}$ (of Point (1) of Assumption 8.5) there exists a family of proper balls B_r such that Assumption 7.1 or Assumption 7.6 holds in B_r (with arbitrarily small τ): basically, we are replacing the assumption of the control of the functionals with the existence of (local) cylinders of approximate flow. The precise assumptions reads as follows:

Assumption 8.12. For all $\tau > 0$

(1) there exist $K^{\tau,\pm}$ compact sets such that
$$\mu^\pm(\mathbb{R}^{d+1} \setminus K^{\tau,\pm}) < \tau;$$
(2) there exists a measure ϖ^τ of mass τ such that
 (a) for all $(t,x) \in K^{\tau,-}$ there exists a family of proper balls $\{B_r^{d+1}(t,x)\}_r$ with 0 as Lebesgue density point and such that Assumption 7.1 or Assumption 7.6 holds forward in $B_r^{d+1}(t,x)$,
 (b) for all $(t,x) \in K^{\tau,+}$ there exists a family of proper balls $\{B_r^{d+1}(t,x)\}_r$ with 0 as Lebesgue density point and such that Assumption 7.1 or Assumption 7.6 holds backward in $B_r^{d+1}(t,x)$,
 (c) for all $(t,x) \in \mathbb{R}^{d+1} \setminus (K^{\tau,-} \cup K^{\tau,+})$ there exists a family of proper balls $\{B_r^{d+1}(t,x)\}_r$ with 0 as Lebesgue density point and such that Assumption 7.1 or Assumption 7.6 holds either backward or forward in $B_r^{d+1}(t,x)$;
(3) it holds $\|\varpi^\tau\| \leq \tau$.

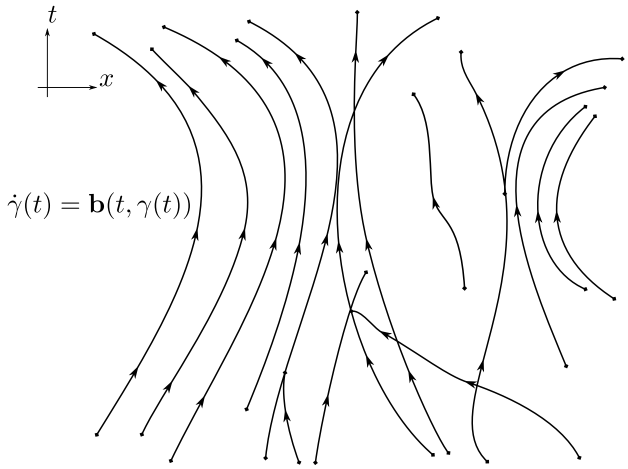
Indeed, for all $(t,x) \in K^-$, by Theorem 7.15 and monotonicity of $\mathfrak{f}^{\mathrm{in}}$, for $\mathcal{L}^1$-a.e. proper balls $B_r^{d+1}(t,x)$ of the family and for all $C > 1$ it holds

$$\mathfrak{f}^{\mathrm{in}}(B_r^{d+1}(t,x)) \leq (C+2)\varpi^\tau(B_r^{d+1}(t,x)) + \frac{\mu^-(B_r^{d+1}(t,x))}{C-1}.$$

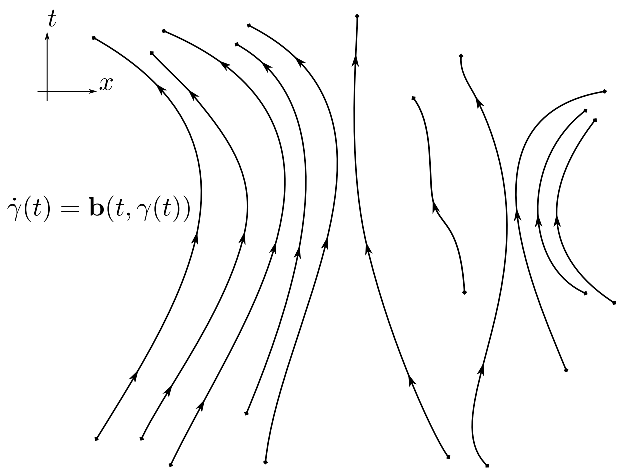
The other cases are completely similar. The choice $C = \tau^{-1/2}$ thus suffices.

9. Partition via characteristics and consequences

In this section we use the assumption that the representation η is untangled to show that a partition of $\mathbb{R}^{d+1}$ made of characteristics $\wp_\alpha$ such that each γ is a subset of these. By disintegrating w.r.t. this partition one can show that the PDE reduces to a one-dimensional ODE with measure r.h.s., and thus a complete description of the solution can be obtained. Moreover, if $\rho' \in L^\infty(\rho\mathcal{L}^{d+1})$ solves $\mathrm{div}(\rho'(1,\mathbf{b})) = \mu'$, then the trajectories of its Lagrangian representation η' are subsets of the same partition $\wp_\alpha$. In particular the explicit form of distribution $\mathrm{div}(\beta(\rho)(1,\mathbf{b}))$ is obtained, settling the Chain Rule Problem.



(a) Initial configuration: the curves ay intersect several times, overlap and bifurcate.



(b) Final configuration: after the untangling, the curves are disjoint, thus forming a partition $\{\wp_{\mathfrak{a}}\}_{\mathfrak{a}}$ of $\mathbb{R}^{d+1}$ up to a set $\rho\mathcal{L}^{d+1}$-negligible.

Figure 9. Visual effect of the *untangling* of trajectories: we start by removing locally a set of curves, whose η measure is controlled, in such a way that the curves are disjoint in a small ball. Iterating this step - thanks to subadditivity - we end up with a family of disjoint, untangled trajectories.

9.1. **Construction of the partition and disintegration.** Let η be an untangled Lagrangian representation and $\varDelta$ a σ-compact set as in Definition 8.10, and consider the following relation $\varDelta$:

$$\gamma \sim \gamma' \quad \Longleftrightarrow \quad \exists N \in \mathbb{N}, \{\gamma_i\}_{i=1}^N \subset \varDelta : \ \Big(\gamma = \gamma_1, \gamma_N = \gamma' \ \wedge \ \sharp\big(\operatorname{Graph}\gamma \cap \operatorname{Graph}\gamma'\big) > 1\Big).$$

It is standard to check that this is an equivalence relation: let $E_{\mathfrak{a}}$, $\mathfrak{a} \in \mathfrak{A}$, be the equivalence classes, being $\mathfrak{A}$ an appropriate set of indexes. Define now $\wp_{\mathfrak{a}}$ as the curve defined in an open interval of time whose graph is

$$\operatorname{Graph}\wp_{\mathfrak{a}} := \bigcup_{\gamma \in E_{\mathfrak{a}}} \operatorname{Graph}\gamma\llcorner_{(t_\gamma^-, t_\gamma^+)}.$$

One can check that $\wp_{\mathfrak{a}}$ is an absolutely continuous curve in $\varGamma$ for every $\mathfrak{a}$ and furthermore it holds

$$\operatorname{Graph}\wp_{\mathfrak{a}} \cap \operatorname{Graph}\wp_{\mathfrak{a}'} = \emptyset$$

for every $\mathfrak{a} \neq \mathfrak{a}'$ (see also Figure 9). We now show that the partition induced by the equivalence classes of this relation is a Borel partition, according to the following

Proposition 9.1. *There exists a Borel map* $\mathtt{f}\colon \mathbb{R}^{d+1} \to \mathbb{R}$ *such that* $\mathtt{f}^{-1}(\mathfrak{a}) = \operatorname{Graph} \wp_{\mathfrak{a}}$.

Proof. It is enough to construct the map restricted to the set of curves $\wp_{\mathfrak{a}}$ whose interval of existence contains a fixed time $\bar{t}$: by repeating the process for a countable set of times one constructs the map in the general case.

The equivalence classes intersecting $A \subset \{t = \bar{t}\}$ can be written as

$$\mathtt{S}(A) = \bigcup_n \mathtt{S}_n(A),$$

where $\mathtt{S}_0(A) = A$ and recursively

$$\mathtt{S}_n(A) = \big\{\gamma \in \Delta : \operatorname{Graph} \gamma \llcorner_{(t^-_\gamma, t^+_\gamma)} \cap \mathtt{S}_{n-1}(A)\big\}.$$

Being the valuation map $\gamma \mapsto e_t(\gamma) = \gamma(t)$ continuous, it follows that each $\mathtt{S}_n(A)$ is Borel if A is Borel, and then the conclusion follows. □

Using again that the evaluation map is Borel, we deduce also

Corollary 9.2. *There exists a Borel map* $\hat{\mathtt{f}}\colon \Delta \to \mathbb{R}$ *such that* $\hat{\mathtt{f}}^{-1}(\mathfrak{a}) = E_{\mathfrak{a}}$.

9.1.1. *Disintegration.* Having at our disposal a partition of the space-time into trajectories, one can try to disintegrate the equation $\operatorname{div}(\rho(1, \mathbf{b})) = \mu$ over this partition obtaining a family of one-dimensional equations: this is the aim of this paragraph.

First, using the fact that $\hat{\mathtt{f}}$ is a Borel map, we can disintegrate η w.r.t. the measure $m := \hat{\mathtt{f}}_\sharp \eta$, so that we write:

$$\eta = \int_{\mathfrak{A}} \eta_{\mathfrak{a}}\, m(d\mathfrak{a})$$

with the property that, for m-a.e. $\mathfrak{a} \in \mathfrak{A}$ the measure $\eta_{\mathfrak{a}}$ is concentrated on $\operatorname{Graph} \wp_{\mathfrak{a}}$. Recall that, by definition of Lagrangian Representation 3.1, it holds

$$\rho \mathcal{L}^{d+1} = \int_\Gamma \Big((\mathrm{id}, \gamma)_\sharp \mathcal{L}^1 \Big)\, \eta(d\gamma), \qquad \mu = \int_\Gamma \big(\delta_{(\mathrm{id},\gamma)(t^-_\gamma)} - \delta_{(\mathrm{id},\gamma)(t^+_\gamma)} \big)\, \eta(d\gamma).$$

Thus, we have

$$\rho \mathcal{L}^{d+1} = \int_{\mathfrak{A}} \bigg[\int_\Gamma \Big((\mathrm{id}, \gamma)_\sharp \mathcal{L}^1 \Big) \eta_\alpha(d\gamma) \bigg] m(d\mathfrak{a}),$$
$$\mu = \int_{\mathfrak{A}} \bigg[\int_\Gamma \big(\delta_{(\mathrm{id},\gamma)(t^-_\gamma)} - \delta_{(\mathrm{id},\gamma)(t^+_\gamma)} \big) \eta_\alpha(d\gamma) \bigg]\, m(d\mathfrak{a}).$$

Using the property that for m-a.e. $\mathfrak{a} \in \mathfrak{A}$ the measure $\eta_{\mathfrak{a}}$ is concentrated on $\operatorname{Graph} \wp_{\mathfrak{a}}$ we have, by Fubini Theorem, for any bounded continuous function φ

$$\begin{aligned}
&\iint_{\mathbb{R}^+ \times \mathbb{R}^d} \varphi(t,x) \rho(t,x)\, \mathcal{L}^{d+1}(dt\, dx) \\
&= \int_{\mathfrak{A}} \bigg[\int_\Gamma \int_{t^-_\gamma}^{t^+_\gamma} \varphi(t, \gamma(t))\, \mathcal{L}^1(dt)\, \eta_{\mathfrak{a}}(d\gamma) \bigg]\, m(d\mathfrak{a}) \\
&= \int_{\mathfrak{A}} \bigg[\iint_{\mathbb{R}^+ \times \Gamma} \varphi(t, \gamma(t)) \mathbb{1}_{(t^-_\gamma, t^+_\gamma)}(t)\, \mathcal{L}^1 \times \eta_{\mathfrak{a}}(dt\, d\gamma) \bigg]\, m(d\mathfrak{a}) \\
&= \int_{\mathfrak{A}} \bigg[\int_{\mathbb{R}^+} \varphi(t, \wp_{\mathfrak{a}}(t)) \bigg(\int_\Gamma \mathbb{1}_{(t^-_\gamma, t^+_\gamma)}(t) \eta_{\mathfrak{a}}(d\gamma) \bigg)\, \mathcal{L}^1(dt) \bigg]\, m(d\mathfrak{a}) \\
&= \int_{\mathfrak{A}} \bigg[\int_{\mathbb{R}^+} \varphi(t, \wp_{\mathfrak{a}}(t)) w_{\mathfrak{a}}(t)\, \mathcal{L}^1(dt) \bigg]\, m(d\mathfrak{a})
\end{aligned}$$

where we have set

$$w_{\mathfrak{a}}(t) := \int_\Gamma \mathbb{1}_{(t^-_\gamma, t^+_\gamma)}(t) \eta_{\mathfrak{a}}(d\gamma) = \eta_{\mathfrak{a}}\big(\big\{ \gamma \in \Gamma : \gamma \text{ is defined in } t, \text{ i.e. } t \in (t^-_\gamma, t^+_\gamma) \big\} \big).$$

Thus, in view of the computation above we have obtained the following decomposition for $\rho\mathcal{L}^{d+1}$:

$$\rho\mathcal{L}^{d+1} = \int_{\mathfrak{A}} (\mathrm{id}, \wp_{\mathfrak{a}})_\sharp (w_{\mathfrak{a}}\mathcal{L}^1)\, m(d\mathfrak{a}). \tag{9.1}$$

In a similar fashion, we define for μ

$$\mu_{\mathfrak{a}} := \int_{\Gamma} \Big[\delta_{(\mathrm{id},\gamma)(t_\gamma^-)} - \delta_{(\mathrm{id},\gamma)(t_\gamma^+)}\Big] \eta_{\mathfrak{a}}(d\gamma),$$

so that

$$\mu = \int_{\mathfrak{A}} \mu_{\mathfrak{a}}\, m(d\mathfrak{a}) \tag{9.2}$$

Notice that the above formula is not a disintegration of μ because the sets of starting and ending points may be not disjoint in general. However, there is no cancellation of mass, since it holds

$$|\mu| = \int_{\mathfrak{A}} |\mu_{\mathfrak{a}}|\, m(d\mathfrak{a}),$$

consequence of the fact that $\mu^\pm$ are orthogonal and η is a Lagrangian representation. By putting together the equation $\mathrm{div}(\rho(1,\mathbf{b})) = \mu$ with the decompositions (9.1) and (9.2), we thus have proved the following

Proposition 9.3. *There exists a measure m on the set $\mathfrak{A}$ such that the decompositions (9.1) and (9.2) hold and*

$$\frac{d}{dt} w_{\mathfrak{a}} = \mu_{\mathfrak{a}}, \qquad \textit{for } m\textit{-a.e. } \mathfrak{a} \in \mathfrak{A}, \tag{9.3}$$

where we consider $w_{\mathfrak{a}}$ extended to 0 outside the domain of $\wp_{\mathfrak{a}}$.

Since it will be useful later, we want to give a special name to the partitions of the space-time on which one can split the equation $\mathrm{div}(\rho(1,\mathbf{b})) = \mu$ as in Proposition 9.3.

Definition 9.4. We will call a Borel map $\mathtt{g}\colon \mathbb{R}^{d+1} \to \mathfrak{A}$ a *partition via characteristics* of $\rho(1,\mathbf{b})\mathcal{L}^{d+1}$ if:

- $\wp_{\mathfrak{a}} := \mathtt{g}^{-1}(\mathfrak{a})$ is a characteristic in some open domain $I_{\mathfrak{a}}$;
- if $\hat{\mathtt{g}}$ denotes the corresponding map $\hat{\mathtt{g}}\colon \Delta \to \mathfrak{A}$, $\hat{\mathtt{g}}(\gamma) := \mathtt{g}(\mathrm{Graph}\,\gamma)$, setting $m := \hat{\mathtt{g}}_\sharp \eta$ and letting $w_{\mathfrak{a}}$ be the disintegration

  $$\rho\,\mathcal{L}^{d+1} = \int_{\mathfrak{A}} (\mathrm{id}, \wp_{\mathfrak{a}})_\sharp (w_{\mathfrak{a}}\mathcal{L}^1)\, m(d\mathfrak{a})$$

  then

  $$\frac{d}{dt} w_{\mathfrak{a}} = \mu_{\mathfrak{a}} \in \mathcal{M}(\mathbb{R}), \qquad \text{for } m\text{-a.e. } \mathfrak{a} \in \mathfrak{A},$$

  where $w_{\mathfrak{a}}$ is considered extended to 0 outside the domain of $\wp_{\mathfrak{a}}$;
- it holds

  $$\mu = \int_{\mathfrak{A}} (\mathrm{id}, \wp_{\mathfrak{a}})_\sharp \mu_{\mathfrak{a}}\, m(d\mathfrak{a}) \qquad \text{and} \qquad |\mu| = \int_{\mathfrak{A}} (\mathrm{id}, \wp_{\mathfrak{a}})_\sharp |\mu_{\mathfrak{a}}|\, m(d\mathfrak{a}).$$

We will say the partition is *minimal* if moreover

$$\lim_{t \to \bar{t}\pm} w_{\mathfrak{a}}(t) > 0 \qquad \forall \bar{t} \in I_{\mathfrak{a}}.$$

Thus, one can rephrase Proposition 9.3 by saying that the map $\mathtt{f}$ is a partition via characteristics of $\rho(1,\mathbf{b})$. Moreover, taking into account the BV regularity of the functions $w_{\mathfrak{a}}$ (for m-a.e. $\mathfrak{a} \in \mathfrak{A}$, in view of (9.3)), we have that $\mathtt{f}$ is also a *minimal* partition via characteristics.

Theorem 9.5. *There exists a minimal partition via characteristics of $\rho(1,\mathbf{b})\mathcal{L}^{d+1}$.*

Proof. From Proposition 9.3, we get $w_{\mathfrak{a}} \in \mathrm{BV}(\mathbb{R})$ for m-a.e. $\mathfrak{a} \in \mathfrak{A}$: hence, we can decompose $\mathbb{R}$ into countably many open intervals $I^n_{\mathfrak{a}} := (t^{n,-}_{\mathfrak{a}}, t^{n,+}_{\mathfrak{a}})$, with $n \in \mathbb{N}$, such that $w_{\mathfrak{a}} > 0$ in each $I^n_{\mathfrak{a}}$ and

$$\lim_{t \to (t^{n,+}_{\mathfrak{a}})-} w_{\mathfrak{a}}(t) = 0 \qquad \text{or} \qquad \lim_{t \to (t^{n,-}_{\mathfrak{a}})+} w_{\mathfrak{a}}(t) = 0.$$

Accordingly, we can define a new partition by further decomposing $\wp_{\mathfrak{a}}$ into countably many curves $\wp^n_{\mathfrak{a}} := \wp_{\mathfrak{a}}|_{I^n_{\mathfrak{a}}}$. By construction, this new partition is again a partition via characteristics of $\rho(1,\mathbf{b})$ and it is indeed minimal. $\square$

9.2. **Uniqueness of partition via characteristics and consequences.** Having proved *existence* of a minimal partition via characteristics of a vector field of the form $\rho(1,\mathbf{b})$, with $\mathrm{div}(\rho(1,\mathbf{b})) = \mu \in \mathcal{M}$, we now face the problem of *uniqueness* of such partition. In this Section, we will show that the partition constructed in Theorem 9.5 is *unique* in a suitable sense, provided every Lagrangian representation of $\rho(1,\mathbf{b})$ is untangled. More precisely, assume that $\rho(1,\mathbf{b})\mathcal{L}^{d+1}$ satisfies Assumption 8.12, and consider $\rho' \in L^\infty(\rho\mathcal{L}^{d+1})$ with

$$\mathrm{div}\,\big(\rho'(1,\mathbf{b})\big) = \mu'.$$

Without loss of generality, being $\rho' \in L^\infty(\rho\mathcal{L}^{d+1})$, we can assume that $|\rho'| \leq \frac{\rho}{2}$ so that

$$\frac{\rho}{2} \leq \rho + \rho' \leq \frac{3\rho}{2}. \tag{9.4}$$

Let η' be a Lagrangian representation of $(\rho+\rho')(1,\mathbf{b})$, which exists because $\rho+\rho' \geq 0$. We now repeat the analysis above considering $(\rho+\rho')(1,\mathbf{b})\mathcal{L}^{d+1}$: notice that, in view of the bounds (9.4), the vector field $(\rho+\rho')(1,\mathbf{b})\mathcal{L}^{d+1}$ still satisfies Point 2 of Assumption 8.12 if $\rho(1,\mathbf{b})\mathcal{L}^{d+1}$ does: indeed, the lateral flux of $\rho+\rho'$ (in Assumption 7.1) is controlled by 3/2 of the lateral flux of ρ.

As before, we thus find a partition of $\mathbb{R}^{d+1}$ (up to a $\rho\mathcal{L}^{d+1}$-null set) into classes $(\tilde{\wp}_\mathfrak{b})_{\mathfrak{b}\in\mathfrak{B}}$. If now we consider the function $u \in L^\infty$ such that $u\rho = \rho+\rho'$ we have

$$\mathrm{div}\,\big(u\rho(1,\mathbf{b})\big) = \mu + \mu' =: \nu.$$

By applying Proposition 9.3 with the classes $\tilde{\wp}_\mathfrak{b}$ we deduce

$$u\rho\,\mathcal{L}^{d+1} = \int (\mathrm{id}, \tilde{\wp}_\mathfrak{b})_\sharp \big(u \circ \tilde{\wp}_\mathfrak{b} w_\mathfrak{b}\, \mathcal{L}^1\big) m(d\mathfrak{b}), \qquad \nu = \int (\mathrm{id}, \tilde{\wp}_\mathfrak{b})_\sharp \nu_\mathfrak{b}\, m(d\mathfrak{b}),$$

and

$$\frac{d}{dt}(u_\mathfrak{b} w_\mathfrak{b}) = \nu_\mathfrak{b}, \qquad \text{where } u_\mathfrak{b} := u \circ \wp_\mathfrak{b}.$$

Notice that the density $w_\mathfrak{b}$ appearing in the disintegration is controlled (up to constants) from below and from above by $w_\mathfrak{a}$ in view of (9.4). This means that the graph of the classes $\wp_\mathfrak{b}$ graph contains the graph of the equivalence relation induced by $\wp_\alpha$, i.e. it has to hold

$$\tilde{\wp}_\mathfrak{b} = N_\mathfrak{b} \cup \bigcup_n \wp_{\mathfrak{a}_n^\mathfrak{b}},$$

where $N_\mathfrak{b}$ is a possibly non-empty closed set. Furthermore, it holds

$$w_\mathfrak{b} = \sum_n w_{\mathfrak{a}_n^\mathfrak{b}} \qquad \text{and} \qquad \mathrm{Tot.Var.}(w_\mathfrak{b}) = \sum \mathrm{Tot.Var.}(w_{\mathfrak{a}_n^\mathfrak{b}})$$

because $\wp_\mathfrak{a}$ is a partition via characteristics. Then since $u_\mathfrak{b} \in L^\infty$ and $w_\mathfrak{b} > 0$ inside $I_{\mathfrak{a}_n^\mathfrak{b}}$, it follows that $u \circ \tilde{\wp}_\mathfrak{b}$ is BV and at the endpoints

$$\liminf_{t\to\bar{t}} |u_\mathfrak{b} w_\mathfrak{b}| \leq \|u\|_\infty \liminf_{t\to\bar{t}} |w_\mathfrak{b}| = 0.$$

Then it is fairly easy to see that

$$\mathrm{Tot.Var.}(u_\mathfrak{b} w_\mathfrak{b}) = \sum_n \mathrm{Tot.Var.}(u_\mathfrak{b} w_{\mathfrak{a}_n^\mathfrak{b}})$$

and thus we conclude with the following universality result.

Theorem 9.6. *If $\rho' \in L^\infty(\rho\mathcal{L}^{d+1})$ then the map $\mathtt{f}$ is a partition via characteristics of $\rho'(1,\mathbf{b})\mathcal{L}^{d+1}$.*

In particular one can deduce that

Corollary 9.7. *The minimal partition of characteristic is unique up to a η-negligible set of trajectories.*

Proof. The set of equivalence classes must be the same up to η-negligible sets, because every representation is untangled. Being the $\mu_\mathfrak{a}$ determined up to m-negligible sets, it follows that $\wp_\mathfrak{a}$ are uniquely determined too, and the in particular the intervals where $w_\mathfrak{a} > 0$. $\square$

9.2.1. *Chain rule.* Using Vol'pert's Chain Rule we obtain the following: for any $\beta \in C^1(\mathbb{R})$ with $\beta(0) = 0$ the distribution

$$\mu_{\mathfrak{a}}^{\beta} := \frac{d}{dt}\big(\beta(u)w_{\mathfrak{a}}\big)$$

is a measure given by

$$\begin{aligned}\mu_{\mathfrak{a}}^{\beta} &:= \sum_{t_i \text{ jump}} \Big[\beta(u_\alpha(t_i^+))w_\alpha(t_i^+) - \beta(u_\alpha(t_i^-))w_\alpha(t_i^-)\Big] + \beta'(u_\alpha)(D^{\mathrm{cont}}u_\alpha)w_\alpha + \beta(u_\alpha)D^{\mathrm{cont}}w_\alpha \\ &= \sum_{t_i \text{ jump}} \Big[\beta(u_\alpha(t_i^+))w_\alpha(t_i^+) - \beta(u_\alpha(t_i^-))w_\alpha(t_i^-)\Big] + \beta'(u_\alpha)(\nu_\alpha)^{\mathrm{cont}} + \big(\beta(u_{\mathfrak{a}}) - u_\alpha\beta'(u_{\mathfrak{a}})\big)\mu_\alpha^{\mathrm{cont}}.\end{aligned} \tag{9.5}$$

A simple computations yields that

$$\|\mu_{\mathfrak{a}}^{\beta}\| \leq \|\beta'\|_\infty \|\nu_{\mathfrak{a}}\| + \|\beta'\|_\infty \|u\|_\infty \|\mu_{\mathfrak{a}}\|.$$

The above estimate allows to conclude with the following proposition.

Proposition 9.8. *For any $\beta \in C^1$ the distribution*

$$\mathrm{div}\,\big(\beta(u)\rho(1, \mathbf{b})\mathcal{L}^{d+1}\big) = \mu^{\beta},$$

where the measure μ^β is given by

$$\mu^{\beta} := \int \mu_{\mathfrak{a}}^{\beta}\, m(d\mathfrak{a}),$$

with $\mu_{\mathfrak{a}}^{\beta}$ defined in (9.5).

In particular, Proposition 9.8 establishes completely the chain rule formula (and, as a consequence, renormalization property) for vector fields $\rho(1, \mathbf{b})$ satisfying Assumption 8.12.

Part 3
The $L^1_{\rm loc}(\mathbb{R};\mathrm{BV}_{\rm loc}(\mathbb{R}^d))$ case

The final part of the paper is devoted to prove that that for $\mathbf{b} \in L^1_t(\mathrm{BV}_x)$ the vector field $(1,\mathbf{b})\mathcal{L}^{d+1}$ satisfies Assumption 8.12, and then it has a minimal partition via characteristic. The construction of the approximate cylinders of flow depends on the local structure of the vector fields, in particular for the singular part of the derivative we strongly rely on the Rank-one Theorem in order to find an ODE describing the main part of $D\mathbf{b}$.

10. A covering of $D^{\rm sing}\mathbf{b}$

The aim of this section it to construct a decomposition of the set where the singular part of the derivative of $\mathbf{b}$ lives into a family of Lipschitz surfaces: we approximate the component of $\mathbf{b}$ in a particular direction with a function whose super-level sets are regular and share essentially a common direction. This will be useful in the following sections to construct the cylinders of approximate flow in the $L^1_t(\mathrm{BV}_x)$ setting.

The decomposition we present here relies essentially on Alberti's Rank-One Theorem (and ultimately on the properties of sets of finite perimeter, in particular the De Giorgi Rectifiability Theorem).

10.1. BV functions and cones.

For $\mathbf{e} \in \mathbb{S}^{d-1}$, $x \in \mathbb{R}^d$ and $0 < a < 1$, let

$$C(\mathbf{e},a;x) := \big\{ y \in \mathbb{R}^d : |(y-x)\cdot \mathbf{e}| \geq a|y-x| \big\}.$$

be the closed, convex cone around $\mathbf{e}$ of vertex x and opening a. We will often think x to be the origin, so we will often write $C(\mathbf{e},a)$ to denote $C(\mathbf{e},a;0)$. The following proposition is well known:

Proposition 10.1. [DL08, Prop. 5.1] *Let $C = C(\mathbf{e},a)$ be a closed convex cone and $v \in \mathrm{BV}(\mathbb{R}^d;\mathbb{R})$. Set*

$$G := \bigg\{ x : \frac{Dv}{|Dv|}(x) \in C \bigg\}.$$

For any closed convex cone $C' := C(\mathbf{e},a')$ with $a' < a$ there exists $w \in \mathrm{BV}(\mathbb{R}^d;\mathbb{R})$ such that $|Dv|\llcorner_G \ll |Dw|$ and

$$\frac{Dw}{|Dw|}(x) \in C' \qquad \text{for } |Dw|\text{-a.e. } x \in \mathbb{R}^d.$$

For our purposes, we need a slight modification of Proposition 10.1. More precisely, we show

Proposition 10.2. *Let $C = C(\mathbf{e},a)$ be a closed convex cone and $v \in \mathrm{BV}(\mathbb{R}^d;\mathbb{R})$. Set*

$$G := \bigg\{ x : \frac{Dv}{|Dv|}(x) \in C \bigg\}.$$

For any closed convex cone $C' := C(\mathbf{e},a')$ with $a' < a$ and for any $\varepsilon > 0$ there exist $\bar r > 0$ and $w \in \mathrm{BV}(\mathbb{R}^d;\mathbb{R})$ such that:

- *$|Dv|\llcorner_G \ll |Dw|$ and*
$$\frac{Dw}{|Dw|}(x) \in C' \qquad \text{for } |Dw|\text{-a.e. } x;$$
- *there exists a family of a'-Lipschitz functions $(L_{i,j})_{i,j\in\mathbb{N}}$ such that, set $E^h_{i,j} := \{L_{i,j} > h\}$, then*
$$|Dw| = \int_{\mathbb{R}} \sum_{i,j} \mathcal{H}^{d-1}\llcorner_{\partial^\star E^h_{i,j}}\, dh.$$
Furthermore, there exist a family of compact sets $(K'_i)_{i\in\mathbb{N}} \subset \mathbb{R}^d$ such that for $r < \bar r$ it holds
$$\bigg| Dv\llcorner_{G_i} - \int_{\mathbb{R}} \sum_{i,j} \nu^h_{i,j}\, \mathcal{H}^{d-1}\llcorner_{E^h_{i,j}}\, dh \bigg| (B^d_r(x)) < \varepsilon |Dv|(B^d_r(x))$$
for every $x \in K'_i$, where $\nu^h_{i,j}(\cdot)$ denotes the outer measure theoretic normal to $E^h_{i,j}$ and $G_i \subset G$ are suitable subsets of G introduced in the proof.

Following [DL08], we decide to present first the proof of Proposition 10.2 in special case, i.e. when v is the characteristic function of a set (which therefore is a set of finite perimeter). This case turns out to be the building block to prove the Proposition in its full generality, via Coarea formula.

10.2. Proof of Proposition 10.2 in the case of a set of finite perimeter.

Proposition 10.3. *Let $C = C(\mathbf{e}, a)$ be a closed convex cone and $E \subset \mathbb{R}^d$ be a set of finite perimeter. Set $v = \mathbb{1}_E$ and*

$$G := \left\{ x : \frac{Dv}{|Dv|}(x) \in C \right\}.$$

For any $a' < a$ and for any $\varepsilon > 0$ there exist $\bar{r} > 0$ and $w \in \mathrm{BV}(\mathbb{R}^d; \mathbb{R})$ such that

- *$|Dv|\llcorner_G \ll |Dw|$ and*
$$\frac{Dw}{|Dw|}(x) \in C' \qquad \text{for } |Dw|\text{-a.e. } x;$$
- *there exist a family of open, C^1 domains $(\Omega_{i,j})_{i,j\in\mathbb{N}} \subset \mathbb{R}^d$ and real non-negative numbers $\lambda_{i,j} \geq 0$ such that*
$$|Dw| = \sum_{i,j} \lambda_{i,j} \mathcal{H}^{d-1}\llcorner_{\partial\Omega_{i,j}}.$$
Furthermore, there exist compact sets $K_i \subset \bigcup_j \partial\Omega_{i,j}$ such that for $r < \bar{r}$ it holds
$$\Big| D\mathbb{1}_E\llcorner_{G_i} - \sum_j \nu_{i,j}\mathcal{H}^{d-1}\llcorner_{\partial^\star\Omega_{i,j}} \Big| \big(B^d_r(x)\big) \leq C_{d-1}\varepsilon |D\mathbb{1}_E|(B^d_r(x))$$
for any $x \in K_i$, where $\nu_{i,j}(\cdot)$ is the outer unit normal to $\Omega_{i,j}$ and $G_i \subset G$ are suitable subsets of G introduced in the proof.

Proof. Let v, E be as in the statement. We denote by $\partial^\star E$ the reduced boundary of E (see Subsection 3.3.1) and let ν be the approximate exterior unit normal to $\partial^\star E$, so that we can write

$$Dv = \nu\mathcal{H}^{d-1}\llcorner_{\partial^\star E}$$

and accordingly the set G is

$$G = \big\{ x \in \partial^\star E : \nu(x) \in C \big\}.$$

Being $\partial^\star E$ rectifiable, in view of Theorem 3.9, we have that G can be decomposed as

$$G = G_0 \cup \bigcup_{i=1}^\infty G_i$$

where:

- $\mathcal{H}^{d-1}(G_0) = 0$ and for $i \geq 1$ each G_i is a subset of a $(d-1)$-dimensional C^1 manifold M_i;
- $\nu|_{G_i}$ coincides with the normal vector $\mathbf{n}_i$ to the manifold M_i.

We now split the argument into steps:

Step 1. For each $i \geq 1$ we claim that there are C^1 open sets $\{\Omega_{i,j}\}_{j\in\mathbb{N}}$ such that, having set $S_{i,j} := \partial\Omega_{i,j}$ the following conditions hold: the exterior normal to $S_{i,j}$ belongs $\mathcal{H}^{d-1}$-a.e. to C' and $\{S_{ij}\}_{j\in\mathbb{N}}$ is a covering of G_i.

Indeed, recall that $C' = C(\mathbf{e}, a')$ and, up to a change of coordinates, we may assume that $\mathbf{e} = \mathbf{e}_d = (0, 0, \dots, 1)$. For any $x \in G_i$, the normal $\mathbf{n}_i(x)$ belongs to $C(\mathbf{e}, a)$, and thus it is transversal to $\mathbf{e}_d^\perp := \mathrm{span}(\mathbf{e}_1, \dots, \mathbf{e}_{d-1})$. This implies that we can choose an open ball $B^d_r(x)$ centered at x such that

$$M_i \cap B^d_r(x) = \big\{ (x^\perp, x) : x = f_i(x^\perp)) \big\}$$

i.e. $M_i \cap B^d_r(x)$ coincides with the graph of a C^1 function $f_i \colon O_i \subset \mathbb{R}^{n-1} \to \mathbb{R}$ where O_i is some bounded open set in $\mathbb{R}^{d-1}$. Moreover, by continuity of the normal $\mathbf{n}_i$, we can choose $B^d_r(x)$ so that $\mathbf{n}_i(y) \in C'$ for every $y \in M_i \cap B^d_R(x)$. By defining

$$\Omega_x := \big\{ (x^\perp, x) : x < f_i(x^\perp)) \big\}$$

then Ω_x turns to be a C^1 open set, the normal to $S_x := \partial\Omega_x$ belongs to the cone C' and S_x covers $B^d_r(x) \cap M_i$. Since we can cover M_i with a countable family of these balls $B^d_r(x)$, the corresponding S_x form the desired countable covering $S_{i,j}$.

Step 2. We now consider the sets $S_{i,j}$. They have all finite $\mathcal{H}^{d-1}$ measure, which we denote by $\ell_{i,j}$ and they cover $\mathcal{H}^{d-1}$-a.e. G. Take any collection $\lambda_{i,j}$ of positive real numbers such that $\sum_{i,j} \lambda_{i,j} \leq 1$ and $\sum_{i,j} \lambda_{i,j}\ell_{i,j} \leq 1$ and finally set

$$w := \sum_{i,j} \lambda_{i,j} \mathbb{1}_{\Omega_{i,j}}.$$

It is immediate to see that w is bounded and of bounded variation since

$$\|w\|_\infty \leq \sum_{i,j} \lambda_{i,j} \leq 1, \qquad |Dw| = \sum_{i,j} \lambda_{i,j} \mathcal{H}^{d-1} \llcorner_{S_{i,j}} \leq 1.$$

For more details, see [DL08].

Step 3. We now exploit some further properties of points in the reduced boundary. Recall that for sets of finite perimeter for every $x \in \partial^\star E$ it holds

$$\lim_{r\to 0} \frac{|D\mathbb{1}_E|(B_r^d(x))}{\omega_{d-1} r^{d-1}} = 1. \tag{10.1}$$

On the other hand, by Lebesgue's differentiation theorem and Area formula, for every $i, j \in \mathbb{N}$, $\mathcal{H}^{d-1}$-a.e. $x \in S_{i,j}$ is a $\mathcal{L}^{d-1}$-density point for the corresponding open set $O_{i,j}$, given by

$$O_{i,j} = \big(\mathrm{id}, f_{i,j}\big)^{-1}(S_{i,j}) \subset \mathbb{R}^{d-1},$$

which explicitly means that

$$\lim_{r\to 0} \frac{\mathcal{L}^{d-1}\big(O_{i,j} \cap B_r^{d-1}(x^\perp)\big)}{\omega_{d-1} r^{d-1}} = 1. \tag{10.2}$$

We now apply Egorov's Theorem to the two limits (10.1), (10.2) (for each i, j): for every $\varepsilon > 0$, there exists $\bar{r} > 0$ and a compact set $F_{i,j}(\varepsilon, \bar{r}) \subset O_{i,j}$, covering $O_{i,j}$ up to a set of $\mathcal{H}^{d-1}$ measure less than ε, such that for any $r < \bar{r}$ it holds

$$\left| \frac{|D\mathbb{1}_E|(B_r^d(f_{i,j}(x^\perp)))}{r^{d-1}} - \omega_{d-1} \right| < \omega_{d-1}\varepsilon, \tag{10.3a}$$

$$\left| \frac{1}{r^{d-1}} \mathcal{L}^{d-1}\big(O_{i,j} \cap B_r^{d-1}(x^\perp))\big) - \omega_{d-1} \right| < \omega_{d-1}\varepsilon \tag{10.3b}$$

for any $x^\perp \in F_{i,j}(\varepsilon, \bar{r})$. We now introduce the following compact set:

$$K_i(\varepsilon, \bar{r}) := \bigcup_{j\in\mathbb{N}} \mathrm{Graph}\,\big(f_{i,j} \llcorner_{F_{i,j}(\varepsilon,\bar{r}))}\big). \tag{10.4}$$

For any $x \in K_i(\varepsilon, \bar{r})$, thanks to (10.3a) it holds for $r < \bar{r}$

$$\omega_{d-1}(1-\varepsilon) r^{d-1} \leq |D\mathbb{1}_E|(B_r^d(x)) \leq \omega_{d-1}(1+\varepsilon) r^{d-1}$$

and, on the other hand, using (10.3b) and being the projection 1-Lipschitz

$$|D\mathbb{1}_E| \llcorner_{G_i} \big(S_{i,j} \cap B_r^d(x)\big) \geq \mathcal{L}^{d-1}\big(O_{i,j} \cap B_r^{d-1}(x^\perp)\big) \geq \omega_{d-1}(1-\varepsilon) r^{d-1}$$

for every j. Thus we get that, for any $x \in K_i(\varepsilon, \bar{r})$ and any $r < \bar{r}$ we have

$$\begin{aligned}
|D\mathbb{1}_E| \llcorner_{G_i} \big(B_r^d(x) \setminus S_{i,j}\big) &\leq |D\mathbb{1}_E|\big(B_r^d(x)\big) - |D\mathbb{1}_E| \llcorner_{G_i} \big(S_{i,j} \cap B_r^d(x)\big) \\
&\leq \omega_{d-1}(1+\varepsilon) r^{d-1} - \omega_{d-1}(1-\varepsilon) r^{d-1} \\
&= 2\varepsilon \omega_{d-1} r^{d-1} \\
&\leq \frac{2\varepsilon}{1-\varepsilon} |D\mathbb{1}_E|(B_r^d(x)).
\end{aligned}$$

To sum up, the set $K_i(\varepsilon, \bar{r})$ is the set of points $x \in \partial^\star E$ for which it holds for $r < \bar{r}$

$$|D\mathbb{1}_E|\big(B_r^d(x) \setminus S_{i,j}\big) \leq C_{d-1}\varepsilon |D\mathbb{1}_E|(B_r^d(x))$$

and

$$\Big| \mathcal{H}^{d-1}\big(B_r^d(x) \cap S_{i,j}\big) - |D\mathbb{1}_E|\big(B_r^d(x)\big) \Big| \leq C_{d-1}\varepsilon |D\mathbb{1}_E|(B_r^d(x)), \tag{10.5}$$

which comes from (10.3b). Finally, by integration of the normal vector, from (10.5), we obtain that for every $x \in K_i(\varepsilon, \bar r)$ and $r < \bar r$ the desired estimate

$$\Big| D\mathbb{1}_{E \llcorner G_i} - \sum_j \nu_{i,j} \mathcal{H}^{d-1} \llcorner_{S_{i,j}} \Big| (B^d_r(x)) \le C_{d-1} \varepsilon |D\mathbb{1}_E|(B^d_r(x))$$

holds and this concludes the proof. □

10.3. **Proposition 10.2 in the general case.** To prove the general case we exploit Coarea formula, as done in [DL08].

Proof. For every $h \in \mathbb{R}$ we consider the function $v_h := \mathbb{1}_{\{v>h\}}$ and, for future reference, we define the measure $\mathscr{M} \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R})$ as $\mathscr{M} := |Dv_h| \otimes \mathcal{L}^1(dh)$, which explicitly means that, for every continuous function $\phi \colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, it holds

$$\begin{aligned} \int_{\mathbb{R}\times\mathbb{R}^d} \phi(x,h)\, \mathscr{M}(dhdx) &= \int \bigg[\int \phi(x,h) \big| D\mathbb{1}_{\{v>h\}} \big|(dx) \bigg] \mathcal{L}^1(dh) \\ &= \int \bigg[\int_{\partial^* \{v>h\}} \phi(x,h)\, \mathcal{H}^d(dx) \bigg] \mathcal{L}^1(dh). \end{aligned}$$

From Coarea formula 3.7 we have that:

- v_h is a BV function for $\mathcal{L}^1$-a.e. h, i.e. $\{v > h\}$ is a set of finite perimeter. Let ν_h be its exterior unit normal;
- it holds
$$\nu_h(x) = \frac{Dv}{|Dv|}(x)$$
for $\mathcal{L}^1$-a.e. $h \in \mathbb{R}$ and $\mathcal{H}^{d-1}$-a.e. $x \in \partial^\star\{v > h\}$, i.e. for $\mathscr{M}$-a.e. (x,h);
- it holds
$$|Dv| = \int_{\mathbb{R}} |Dv_h| \mathcal{L}^1(dh);$$
- it holds $\mathtt{p}_{\mathbb{R}^d}\big(\mathscr{M}\big) = |Dv|$, hence $\mathscr{M}$ can be disintegrated as
$$\mathscr{M} = \int \mathscr{M}_x |Dv|(dx), \tag{10.6}$$

Therefore, for $\mathcal{L}^1$-a.e. h we can apply Proposition 10.3. We denote by w_h the corresponding bounded, BV function given by Proposition 10.3 and we set

$$w(x) := \int_{\mathbb{R}} w_h(x)\, dh. \tag{10.7}$$

Notice that, in order to write (10.7), we have to be sure that the map $h \mapsto w_h$ enjoys some measurability properties. To show the existence of such a selection, one can use the Aumann Measurable Selection Theorem (for the precise argument we refer again the reader to [DL08]). Then it is immediate to see that w satisfies $|Dv|\llcorner_G \ll |Dw|$ and

$$\frac{Dw}{|Dw|}(x) \in C' \qquad \text{for } |Dw|\text{-a.e. } x.$$

Furthermore, denoting by $S^h_{i,j}$ and $K^h_i(\varepsilon, \bar r)$ the corresponding sets for w_h (obtained via Proposition 10.3), we have that for any $x \in K^h_i(\varepsilon, \bar r)$ for $r < \bar r$ it holds

$$|Dv_h|\big(B^d_r(x) \setminus S^h_{i,j}\big) \le C_{d-1} \varepsilon |Dv_h|(B^d_r(x))$$

and

$$\Big| \mathcal{H}^{d-1}\big(B^d_r(x) \cap S^h_{i,j}\big) - |Dv_h|\big(B^d_r(x)\big) \Big| \le C_{d-1}\varepsilon |Dv_h|(B^d_r(x)).$$

By means of measurable selection, we can define now the measurable sets

$$\widetilde K_i(\varepsilon, \bar r) := \{(x,h) : x \in K^h_i(\varepsilon, \bar r)\} \subset \mathbb{R}^d \times \mathbb{R}, \qquad i \in \mathbb{N}$$

so that for every h we have $\widetilde{K}_i(\varepsilon,\bar r;h) = K_i^h(\varepsilon,\bar r)$; observe that, by construction, they cover $\mathbb{R}^d\times\mathbb{R}$ up to a set of $\mathscr{M}$ - measure less than ε. In view of the disintegration (10.6) we thus can write for all $R>0$

$$\int_{B_R^d(0)} \mathscr{M}_x\Big((\mathbb{R}^d\times\mathbb{R})\setminus \widetilde{K}_i(\varepsilon,\bar r)\Big)|Dv|(dx) < \varepsilon.$$

Thus, by Chebyshev inequality, we deduce that $\widetilde{K}_i(\varepsilon,\bar r)$ covers almost all the fiber of an arbitrary large fraction of points x (in any ball $B_R^d(0)$)): in other words, for every fixed $\delta>0$, there is a set $N_\delta^i\subset B_R^d(0)$ such that

$$|Dv|(N_\delta^i) < \frac{\varepsilon}{1-\delta}$$

and

$$\mathscr{M}_x\Big(\widetilde{K}_i(\varepsilon,\bar r)\Big) > 1-\delta, \qquad \forall x\in B_R^d(0)\setminus N_\delta^i. \tag{10.8}$$

Taking a compact set $K_i'\subset B_R^d(0)\setminus N_\delta^i\subset \mathtt{p}_{\mathbb{R}^d}(\widetilde{K}_i(\varepsilon,\bar r))$ we obtain that for every $x\in K_i'$ and $r<\bar r$ it holds

$$\big|Dw\llcorner_{G_i} - Dv\big|\big(B_r^d(x)\big)\le C_{d-1}\varepsilon|Dv|(B_r^d(x)),$$

which is the claim. □

It is now clear that we can repeat finitely many times the above constructions in order to cover all the reduced boundary. More precisely, given any $\delta_c>0$, we pick a set of unit vectors $\{\mathbf{n}_s, s=1,\dots,J_{\delta c}\}\subset\mathbb{R}^d$ in such a way that

$$B_1^d(0)\subset\bigcup_{s=1}^{J_{\delta_c}} C(\mathbf{n}_s,\delta_c).$$

By choosing $a'=\delta_c/2$ and applying Proposition 10.2, we obtain the following

Corollary 10.4. *Let $v\in \mathrm{BV}(\mathbb{R}^d;\mathbb{R})$ and for every $s=1,\dots,J_{\delta_c}$ set*

$$G_s := \Big\{x:\ \frac{Dv}{|Dv|}(x)\in C(\mathbf{n}_s,\delta_c)\Big\}.$$

For every $\varepsilon>0$ there exist $\bar r>0$ and $w\in\mathrm{BV}(\mathbb{R}^d;\mathbb{R})$ such that:

- *$|Dv|\llcorner_{G_s}\ll|Dw|$ for every $s=1,\dots,J_{\delta c}$ and*

$$\frac{Dv}{|Dv|}(x)\in C(\mathbf{n}_s,\delta_c)\Rightarrow\frac{Dw}{|Dw|}(x)\in C\Big(\mathbf{n}_s,\frac{\delta_c}{2}\Big)\qquad \text{for } |Dw|\text{-a.e. } x;$$

- *for every $s=1,\dots,J_{\delta c}$ there exists a family of C^1 functions $(L_{i,j,s})$ for $i,j\in\mathbb{N}$, with Lipschitz constant δ_c, such that, setting $E_{i,j,s}^h := \{L_{i,j,s}>h\}$, then*

$$|Dw|\llcorner_{G_s} = \int_{\mathbb{R}}\sum_{i,j}\mathcal{H}^{d-1}\llcorner_{\partial^\star E_{i,j,s}^h}\,dh.$$

Furthermore, there exists a family of compact sets $(K_{s,i}')\subset\mathbb{R}^d$ with $i\in\mathbb{N}$ and $s\in\{1,\dots,J_{\delta c}\}$ such that for $r<\bar r$ it holds

$$\bigg|Dv\llcorner_{G_{s_i}} - \int_{\mathbb{R}}\sum_{i,j}\nu_{j,s}\mathcal{H}^{d-1}\llcorner_{\partial^\star E_{i,j,s}^h}\,dh\bigg|(B_r^d(x)) < \varepsilon|Dv|(B_r^d(x))$$

for every $x\in K_{s,i}'$, where $\nu_{i,j,s}^h(\cdot)$ is the outer unit normal to $E_{i,j,s}^h$ and $G_{s_i}\subset G_s$ are suitable subsets of G_s.

10.4. **Decomposition for vector fields $L^1_{\rm loc}(\mathrm{BV}_{\rm loc}(\mathbb{R}^d,\mathbb{R}^d))$.** We now consider the vector-valued case, i.e. we take $\mathbf{b}\in L^1_{\rm loc}(\mathbb{R},\mathrm{BV}_{\rm loc}(\mathbb{R}^d,\mathbb{R}^d))$ and we are interested in covering the singular part of $D\mathbf{b}$: in order to achieve this, we have to exploit Alberti's Rank one Theorem 3.8.

More precisely, let us denote by n,m the two unit vectors given by Rank one property, i.e. such that

$$D^{\rm sing}\mathbf{b}=\mathsf{m}\otimes\mathsf{n}|D^{\rm sing}\mathbf{b}|.$$

Consider the points $(\bar t,\bar x)$ with the following properties:

- $(\bar t,\bar x)$ is a point where the measure $D\mathbf{b}$ is essentially singular, i.e. it is a density point for $D^{\rm sing}\mathbf{b}$. More precisely, $(\bar t,\bar x)$ is such that for every $\varepsilon>0$ there exists $\bar r(\varepsilon,\bar t,\bar x)>0$ such that for $0<r<\bar r$ it holds
$$|D^{\rm sing}\mathbf{b}|(B^{d+1}_r(\bar t,\bar x))>(1-\varepsilon)|D\mathbf{b}|(B^{d+1}_r(\bar t,\bar x));\tag{10.9}$$
- $(\bar t,\bar x)$ is a Lebesgue point of the matrix valued map $(t,x)\mapsto\mathsf{m}\otimes\mathsf{n}(t,x)$, which is defined $|D^{\rm sing}\mathbf{b}|$-a.e., that is to say for every $\varepsilon>0$ there exists $\bar r'(\varepsilon,\bar t,\bar x)>0$ such that for $0<r<\bar r'$ it holds
$$\int_{B^{d+1}_r(\bar t,\bar x)}|\mathsf{m}\otimes\mathsf{n}-\bar{\mathsf{m}}\otimes\bar{\mathsf{n}}||D^{\rm sing}\mathbf{b}|(dtdx)<\varepsilon|D^{\rm sing}\mathbf{b}|(B^{d+1}_r(\bar t,\bar x)),\tag{10.10}$$
having denoted by $\bar{\mathsf{m}}\otimes\bar{\mathsf{n}}$ the Lebesgue value in $(\bar t,\bar x)$.

By a standard application of Egorov Theorem, for every fixed $\varepsilon>0$ we can find a sequence $(\bar r_i)_{i\in\mathbb{N}}$ (where $\bar r_i$ depend only on ε) and a family of compact sets $\big(G(\varepsilon,r_i)\big)_{i\in\mathbb{N}}\subset\mathbb{R}^{d+1}$ covering almost all the set where $D^{\rm sing}\mathbf{b}$ is concentrated and such that the limits (10.9) and (10.10) are uniform on each $G(\varepsilon,r_i)$. Moreover, we can further split the compact sets $G(\varepsilon,r_i)$ according to the direction: indeed, we denote by $G(\varepsilon,r_i,s)$ the set of points $(t,x)\in G(\varepsilon,r_i)$ such that $\mathsf{m}(t,x)\in C(\mathbf{n}_s,\delta_c)$, for $s\in\{1,\dots,J_{\delta_c}\}$.

Now, we denote by $b_{\mathsf{n}}:=\mathbf{b}\cdot\bar{\mathsf{n}}$ the component of $\mathbf{b}$ along $\bar{\mathsf{n}}$. By Rank one, the (scalar) function $b_{\bar{\mathsf{n}}}$ has polar vector $\bar{\mathsf{m}}$ in $(\bar t,\bar x)$. Thus, by Chebyshev inequality, we can say that for an arbitrary large fraction (w.r.t $D^{\rm sing}\mathbf{b}$) of points $(t,x)\in G(\varepsilon,r_i,s)$ it holds

$$\frac{Db_{\mathsf{n}}}{|Db_{\mathsf{n}}|}(t,x)=\mathsf{m}\in C(\mathbf{n}_s,\delta_c)$$

since m is close to $\bar{\mathsf{m}}$ in view of (10.10). Therefore, we are in position to apply Corollary 10.4: there are a BV function $\mathcal{U}_{\bar{\mathsf{n}}}$ and C^1 functions (with Lipschitz constant less than δ_c) $(L^{\bar{\mathsf{n}}}_{i,j,s})$ for $i,j\in\mathbb{N}$ and $s\in\{1,\dots,J_{\delta_c}\}$ such that, set $E^{\bar{\mathsf{n}},h}_{i,j,s}:=\{L^{\bar{\mathsf{n}}}_{i,j,s}>h\}$, then the derivative of $\mathcal{U}_{\bar{\mathsf{n}}}$ can be written as

$$|D\mathcal{U}_{\bar{\mathsf{n}}}|\llcorner_{G_s}=\int_{\mathbb{R}}\sum_{i,j}\mathcal{H}^{d-1}\llcorner_{\partial^\star E^{s,\bar{\mathsf{n}},h}_{i,j}}\,dh.$$

Furthermore, there exist $\bar r>0$ and a family of sets $(K^{\bar{\mathsf{n}}}_s)_s\subset\mathbb{R}\times\mathbb{R}^d$ such that for $r<\bar r$ it holds

$$\left|Db_{\mathsf{n}}-\int_{\mathbb{R}}\sum_{i,j}\nu^{\bar{\mathsf{n}},h}_{i,j,s}\mathcal{H}^{d-1}\llcorner_{\partial^\star E^{\bar{\mathsf{n}},h}_{i,j,s}}\,dh\right|(B^d_r(x))<\varepsilon|Db_{\mathsf{n}}|(B^d_r(x))$$

for every $x\in K^{\bar{\mathsf{n}}}_s$ where $\nu^{\bar{\mathsf{n}},h}_{i,j,s}$ is the outer unit normal to $E^{\bar{\mathsf{n}},h}_{i,j,s}$.

Finally if we multiply back times $\bar{\mathsf{m}}$ we end up with a matrix valued measure which is the derivative of an approximated BV vector field: this yields a sort of vectorial analog of Corollary 10.4. By expliciting the normal to the set $E^{\bar{\mathsf{n}},h}_{i,j,s}$, observing that the map $(t,x,h)\mapsto\mathbb{1}_{\{\mathbf{b}_t\cdot\bar{\mathsf{n}}>h\}}(x)$ is measurable and using again a measurable selection argument, we can finally state the following

Corollary 10.5. *Let $\mathbf{b}\in L^1_{\rm loc}(\mathbb{R},\mathrm{BV}_{\rm loc}(\mathbb{R}^d,\mathbb{R}^d))$. Then for every $\varepsilon>0$ and $\delta_c>0$ there exists compact sets $K^{\varepsilon,j}_{\delta_c,r_i}$ the such that if $(\bar t,\bar x)\in K^{\varepsilon,j}_{\delta_c,r_i}$ then there exist a family of C^1 functions $\{y_{\mathbf{n}_j}=L_{t,h}(y^\perp_{\mathbf{n}_j})\}_{t,h}$ with Lipschitz constant less than δ_c such that*

$$\left|D\mathbf{b}-\int\Big\{\big[\mathsf{m}(\bar t,\bar x)\otimes(1,-\nabla_{y^\perp_{\bar{\mathsf{n}}}}L_{t,h})\big]\,\delta_t\otimes\big((\mathrm{id},L_{t,h})_\sharp\mathcal{L}^{d-1}\big)\Big\}\,dtdh\right|\big(B^{d+1}_r(\bar t,\bar x)\big)<C\varepsilon|D^{\rm sing}\mathbf{b}|\big(B^{d+1}_r(\bar t,\bar x)\big).$$

11. Construction of approximate cylinders of flow in the BV setting

The aim of this section is to construct locally some approximate flow cylinders, which maintain a quite regular shape and have a small boundary flow. We want to verify that Assumption 7.1 or Assumption 7.6 holds in a neighborhood of every point (t,x), and that then Assumption 8.12 is valid.

As observed in Remark 4.6, one has to control the lateral flow either for a family of smooth Lipschitz functions ϕ^ℓ_γ or Lipschitz sets Q^ℓ_γ, the two conditions being equivalent.

11.1. Estimates for the absolutely continuous part.

Fix a matrix A. For $\gamma \in \Gamma$ define the cylinder

$$\phi^{\ell,\delta_1}_\gamma\big(t,\gamma(t)+e^{At}y\big) = \bigg[1 - \frac{1}{\delta_1\ell}\operatorname{dist}\big(y, B^d_\ell(0)\big)\bigg]^+,$$

and the normalization constant

$$\sigma^{\ell,\delta_1} = \int \bigg[1 - \frac{1}{\delta_1\ell}\operatorname{dist}\big(x, B^d_\ell(0)\big)\bigg]^+ \mathcal{L}^d(dx).$$

A standard computation gives

$$\begin{aligned}
&\frac{1}{\sigma^{\ell,\delta_1}}\int_{t^-_\gamma}^{t^+_\gamma}\int \big|(1,\mathbf{b})\cdot\nabla_{t,x}\phi^{\ell,\delta_1}_\gamma\big|\,\mathcal{L}^{d+1} \\
&\quad= \frac{1}{\sigma^{\ell,\delta_1}}\int_{t^-_\gamma}^{t^+_\gamma}\Big| -\nabla_x\phi^{\ell,\delta_1}_\gamma\cdot\big(\mathbf{b}(t,\gamma(t)) + Ae^{At}y\big) + \mathbf{b}(t,x)\cdot\nabla_x\phi^{\ell,\delta_1}_\gamma(t,x)\Big|\,\mathcal{L}^{d+1} \\
&\quad= \int_{t^-_\gamma}^{t^+_\gamma}\frac{1}{\delta_1\ell}\int_{|y|\in\ell(1,1+\delta_1)}\bigg|\Big(\mathbf{b}(t,\gamma(t)+e^{At}y) - \mathbf{b}(t,\gamma(t)) - Ae^{At}y\Big)\cdot e^{-At}\frac{y}{\sigma^{\ell,\delta_1}|y|}\bigg| e^{\mathrm{tr}At}\,\mathcal{L}^{d+1}(dy),
\end{aligned}$$

so that if η is a Lagrangian representation for $(1,\mathbf{b})\mathcal{L}^{d+1}\llcorner\Omega$ with Ω Lipschitz (because of Proposition 5.12)

$$\begin{aligned}
&\int \frac{1}{\sigma^{\ell,\delta_1}}\bigg[\int_{t^-_\gamma}^{t^+_\gamma}\int \big|(1,\mathbf{b})\cdot\nabla_{t,x}\phi^{\ell,\delta_1}_\gamma\big|\,\mathcal{L}^{d+1}\bigg]\eta(d\gamma) \\
&\quad\le \int\bigg[\int_{t^-_\gamma}^{t^+_\gamma}\frac{1}{\sigma^{\ell,\delta_1}\delta_1\ell}\int_{|y|\in\ell(1,1+\delta_1)}\Big|\mathbf{b}\big(t,\gamma(t)+e^{At}y\big) - \mathbf{b}(t,\gamma(t)) - Ae^{At}y\Big|\frac{|e^{-At}y|}{|y|}e^{\mathrm{tr}At}\,\mathcal{L}^{d+1}(dtdy)\bigg]\eta(d\gamma) \\
&\quad\le \frac{1}{\delta_1\omega_d\ell^{d+1}}\int_{|e^{-At}z|\in\ell(1,1+\delta_1)}\frac{|e^{-2At}z|}{|e^{-At}z|}\int_\Omega\big|\mathbf{b}(t,x+z) - \mathbf{b}(t,x) - Az\big|\,\mathcal{L}^{2d+1}(dtdxdz) \\
&\quad\le \frac{1}{\delta_1\omega_d\ell^{d+1}}\int_{|e^{-At}z|\in\ell(1,1+\delta_1)}\frac{|e^{-2At}z|}{|e^{-At}z|}|z|\big|D\mathbf{b}_t - A\mathcal{L}^d\big|\big(\Omega_t + B^d_{(1+\delta_1)e^{\|A\|t}\ell}(0)\big)\,\mathcal{L}^{d+1}(dtdx) \\
&\quad\le C_d\|e^{2\|A\|t}\|_{L^\infty(\mathtt{p}_t\Omega)}\big|D\mathbf{b} - A\mathcal{L}^{d+1}\big|\big(\Omega + \{t=0\}\times B^d_{(1+\delta_1)\|e^{\|A\|t}\|_{L^\infty(\mathtt{p}_t\Omega)}\ell}(0)\big).
\end{aligned} \tag{11.1}$$

Letting $\ell,\delta_1\to 0$ and choosing the matrices A in order to approximate the a.c. part of $D\mathbf{b}$, we conclude with the following proposition.

Proposition 11.1. *For every point (t,x) there exists $\bar r_{t,x}$ such that for $\mathcal{L}^1$-a.e. $0<r<\bar r_{t,x}$ the ball $B^{d+1}_r(t,x)$ is $(1,\mathbf{b})$-proper and Assumption 7.1 holds with constant $\varpi_r(t,x)$ such that*

$$\varpi_r(t,x) \le \begin{cases} \tau|D\mathbf{b}|(B^{d+1}_r(t,x)) & (t,x)\ \text{Lebesgue point for } |D^{\mathrm{a.c.}}\mathbf{b}|, \\ C_d|D\mathbf{b}|(B^{d+1}_r(t,x)) & \text{otherwise.}\end{cases}$$

The proof is just an application of the Radon-Nikodym theorem, and it will be omitted.

11.2. Estimates for the singular part.

Fix $0<\tau\ll 1$, and set

$$\delta_c = \frac{\tau^2}{2}$$

By Corollary 10.5 of Section 10, there exists a compact set $K^\tau_{\delta_c,\bar r}$ such that

(1) its complement has small measure

$$|D^{\rm sing}\mathbf{b}|\big(\mathbb{R}^{d+1}\setminus K^{\tau}_{\delta_c,\bar r}\big)<\tau;$$

(2) each $(\bar t,\bar x)\in K^{\tau}_{\delta_c,\bar r}$ is a Lebesgue point for $\mathsf{m}\otimes\mathsf{n}$: denote by

$$\overline{\mathsf{m}}\otimes\overline{\mathsf{n}}=\mathsf{m}\otimes\mathsf{n}(\bar t,\bar x)$$

its value, and for every $r<\bar r$ it holds

$$|D^{\rm a.c.}\mathbf{b}|(B^{d+1}_r(\bar t,\bar x)),\int_{B^{d+1}_r(\bar t,\bar x)}|\mathsf{m}\otimes\mathsf{n}-\overline{\mathsf{m}}\otimes\overline{\mathsf{n}}||D^{\rm sing}\mathbf{b}|(dtdx)<\tau^2|D^{\rm sing}\mathbf{b}|(B^{d+1}_r(\bar t,\bar x));\tag{11.2}$$

(3) for every $(\bar t,\bar x)\in K^{j,\tau}, r<\bar r$ there exists a compact family of δ_c-Lipschitz functions $\{y_{\overline{\mathsf{n}}}=L_{t,h}(y^{\perp}_{\overline{\mathsf{n}}})\}_{h\in H}$ such that defining the function $\mathcal{U}$ by

$$\mathcal{U}(t,\bar x)=0,\quad D\mathcal{U}(t)=\int_H\Big\{\big[\overline{\mathsf{m}}\otimes(1,-\nabla_{y^{\perp}_{\overline{\mathsf{n}}}}L_{t,h})\big]\,\delta_t\otimes\big((\mathrm{id},L_{t,h})_\sharp\mathcal{L}^{d-1}\big)\Big\}\,dtdh,$$

then it holds

$$|D\mathcal{U}-D\mathbf{b}|(B^{d+1}_r(\bar t,\bar x))<\tau^2|D\mathbf{b}|(B^{d+1}_r(\bar t,\bar x)).\tag{11.3}$$

This compact set is obtained by the union of the compact sets $K^{\tau,j}_{\delta_c,\bar r}$ of Corollary 10.5, with $\bar r\ll 1$.

11.2.1. *Construction of the approximate cylinders of flow.* We can assume that $\overline{\mathsf{n}}=\mathbf{e}_1$, and write $y=(y_1,y^\perp)\in\mathbb{R}\times\mathbb{R}^{d-1}$ for the corresponding coordinates. Set

$$\bar\ell_1:=\tau\ell,\qquad \delta_1:=\tau^2,\qquad \ell>0,\tag{11.4}$$

and let η be a Lagrangian representation of $\rho(1,\mathbf{b})\mathcal{L}^{d+1}\llcorner_{B^{d+1}_{\bar r}(\bar t,\bar x)}$.

We consider three cases.

Case 1: $\overline{\mathsf{m}}_1=\overline{\mathsf{m}}\cdot\overline{\mathsf{n}}=\overline{\mathsf{m}}\cdot\mathbf{e}_1<-\tau$. For every $\gamma\in\Gamma$, define the functions $\ell^{\pm}_{1,\gamma}:[t^-_\gamma,t^+_\gamma]\times B^{d-1}_\ell\to\mathbb{R}$ by solving the following ODEs:

$$\partial_t\ell^-_{1,\gamma}(t,y^\perp)=-\mathcal{U}_1\big(t,\gamma(t)+(-\ell^-_{1,\gamma}(t,y^\perp)+,y^\perp)\big)+\mathcal{U}_1\big(t,\gamma(t)+((-\delta_1-\delta_c)\ell+,0)\big),\tag{11.5a}$$

$$\partial_t\ell^+_{1,\gamma}(t,y^\perp)=\mathcal{U}_1\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t,y^\perp)-,y^\perp)\big)-\mathcal{U}_1\big(t,\gamma(t)+((\delta_1+\delta_c)\ell-,0)\big),\tag{11.5b}$$

with initial data $\ell^{\pm}_{1,\gamma}(t^-_\gamma,y^\perp)=\bar\ell_1$. We recall that $\mathcal{U}_1=\mathcal{U}\cdot\mathbf{e}_1=\mathcal{U}\cdot\overline{\mathsf{n}}$, and we have denoted with $\pm$ the right/left limits of 1-d BV functions.

Lemma 11.2. *The solutions to* (11.5) *satisfy*

(1) $[t^-_\gamma,t^+_\gamma]\ni t\mapsto\ell^{\pm}_{1,\gamma}(t,y^\perp)$ *is decreasing;*
(2) $B^{d-1}_\ell(0)\ni y^\perp\mapsto\ell^{\pm}_{1,\gamma}(t,y^\perp)$ *is δ_c-Lipschitz continuous;*
(3) $\delta_1\ell\leq\ell^{\pm}_{1,\gamma}(t,y^\perp)\leq\bar\ell_1$ *for all* $(t,y^\perp)\in[t^-_\gamma,t^+_\gamma]\times B^{d-1}_\ell(\bar t,\bar x)$.

Proof. We prove the lemma for $\ell^+_{1,\gamma}$, being the analysis of $\ell^-_{1,\gamma}$ equivalent. The existence of a unique solution which is decreasing in time is standard, see for example [BG11]: indeed for fixed $(t,y^\perp)$

$$y_1\mapsto\mathcal{U}_1(t,(y_1,y^\perp))$$

is decreasing because $\overline{\mathsf{m}}_1<0$, and then classical results on the flow of monotone operators apply.

The fact that the level sets of $\mathcal{U}_1$ are δ_c-Lipschitz in the coordinates $(y_1,y^\perp)$ implies that

$$\mathcal{U}_1\big(t,\gamma(t)+(\delta_1\ell+\delta_c(\ell-|y^\perp|)-,y^\perp)\big)\geq\mathcal{U}_1\big(t,\gamma(t)+((\delta_1+\delta_c)\ell-,0)\big),$$

so that the solution starting from $\bar\ell_1=\tau\ell>(\delta_1+\delta_c)\ell$ satisfies

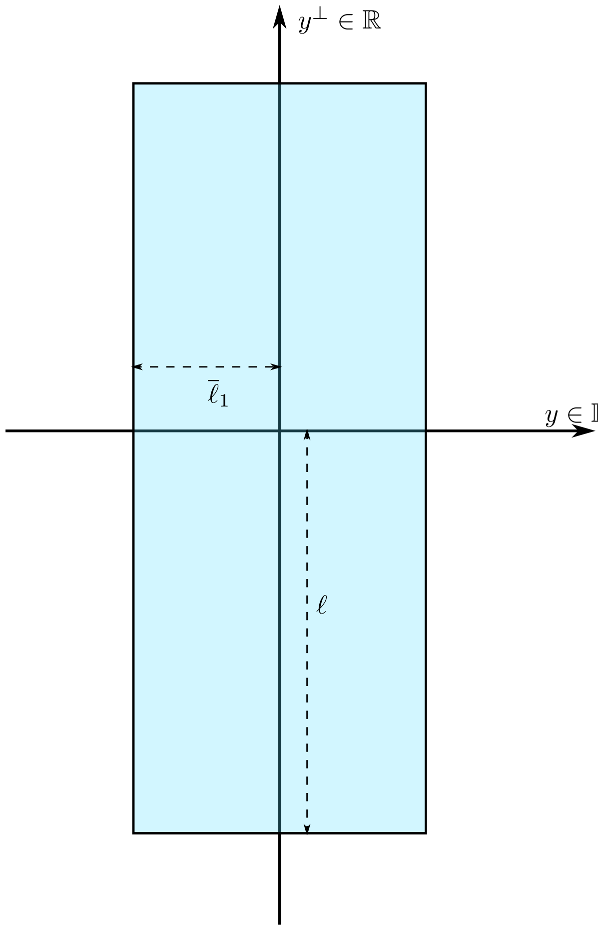
$$\ell^+_{1,\gamma}(t,y^\perp)\geq\delta_1\ell+\delta_c(\ell-|y^\perp|)\geq\delta_1$$

when $|y^\perp|<\ell$.

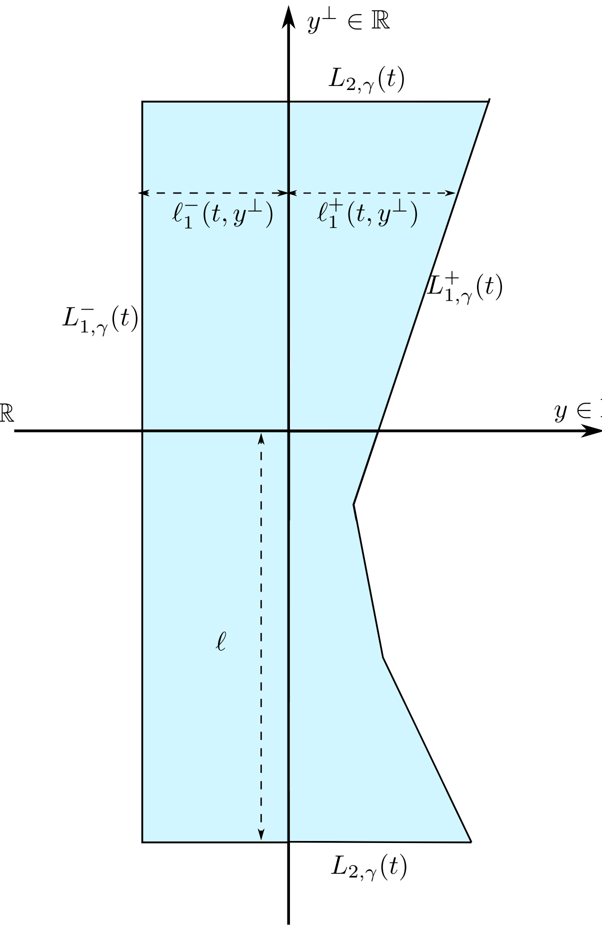
For $\bar y^\perp$ fixed, again from the δ_c-Lipschitz regularity of the level sets of $\mathcal{U}$, it is easy to see that the cone

$$\big|y_1-\ell^{\pm}_{1,\gamma}(t,\bar y^\perp)\big|\leq\delta_c|y^\perp-\bar y^\perp|$$

is invariant for the flow of the ODEs (11.5), so that for any fixed time t it holds that $\ell^+_{1,\gamma}(t,y^\perp)$ is δ_c-Lipschitz. $\square$



(a) The base of the cylinder, i.e. the set $\bar{Q}$.



(b) The base of the cylinder at a time t, i.e. the set $Q(t)$.

Figure 10. Time sections of the cylinder of approximate flow in the singular, 2D case.

Case 2: $\overline{\mathrm{m}}_1 > \tau$**.** Define the functions $\ell^\pm_{1,\gamma} : [t^-_\gamma, t^+_\gamma] \times B^{d-1}_\ell \to \mathbb{R}$ by solving the ODEs (11.5a) backward in time with final data $\ell^\pm_{1,\gamma}(t^+_\gamma, y^\perp) = \bar{\ell}_1$. As in Lemma 11.2, one can check that

(1) $[t^-_\gamma, t^+_\gamma] \mapsto \ell^\pm_{1,\gamma}(t, y^\perp)$ is increasing;
(2) $B^{d-1}_r(0) \ni y^\perp \mapsto \ell^\pm_{1,\gamma}(t, y^\perp)$ is δ_c-Lipschitz continuous;
(3) $\delta_1 \ell \le \ell^\pm_{1,\gamma}(t, y^\perp) \le \bar{\ell}_1$.

Case 3: $|\overline{\mathbf{m}}_1| < \tau$**.** In this case set $\ell^\pm_{1,\gamma}(t) = \bar{\ell}_1$ constant.

Define (see Figure 10b and 11b)

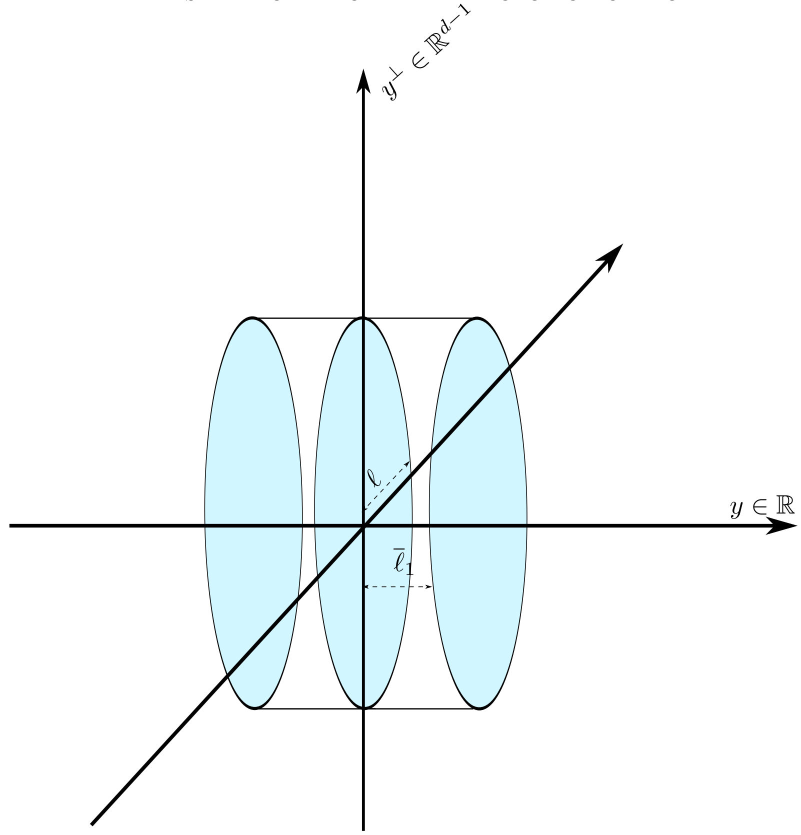
$$Q^\ell_\gamma(t) = Q_{\ell^\pm_{1,\gamma},\ell} = Q_{\ell^-_{1,\gamma},\ell^+_{1,\gamma},\ell}(t) := \Big\{ y = (y_1, y^\perp) : -\ell^-_1(t, y^\perp) \le y \le \ell^+_1(t, y^\perp),\, |y^\perp| \le \ell \Big\}.$$

For future reference we call

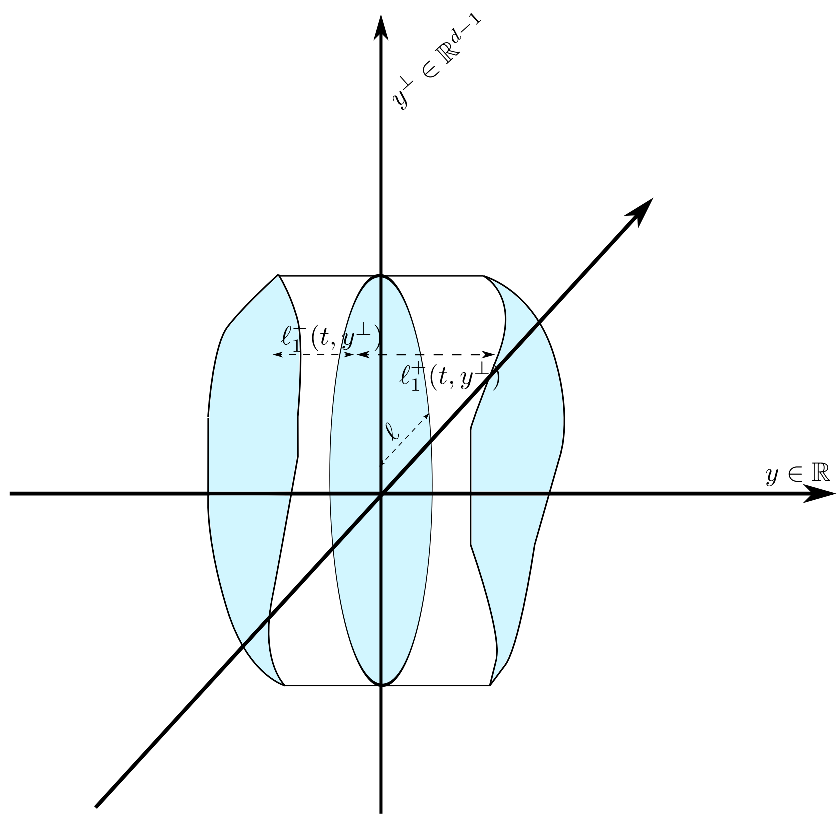
$$\bar{Q} := \Big\{ y = (y, y^\perp) : -\bar{\ell}_1 \le y \le \bar{\ell}_1,\, |y^\perp| \le \ell \Big\},$$

see also Figure 10a and 11a. Define the lateral sides of $Q_{\ell^\pm_{1,\gamma}\ell}(t)$ as

$$L^\pm_{1,\gamma}(t) := \pm \operatorname{Graph} \ell^\pm_{1,\gamma}(t), \qquad L_{2,\gamma}(t) = \Big\{ (y_1, y^\perp), -\ell^-_{1,\gamma}(t, y^\perp) \le y_1 \le \ell^+_{1,\gamma}(t, y^\perp), |y^\perp| = \ell \Big\}.$$



(a) The base of the cylinder, i.e. the set $\bar{Q}$.



(b) The base of the cylinder at a time t, i.e. the set $Q(t)$.

Figure 11. Time sections of the cylinder of approximate flow in the singular, d-dimensional case.

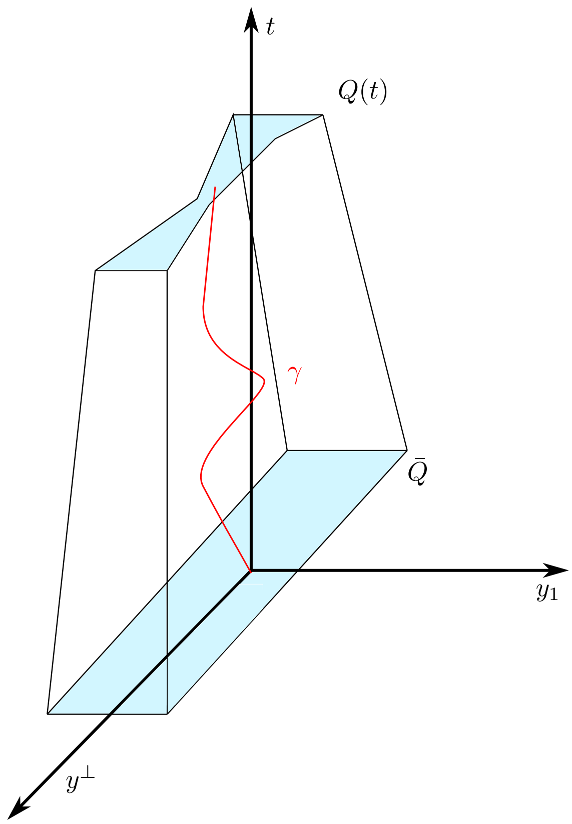


Figure 12. Evolution in time of the cylinder of approximate flow in the singular, 2D case.

After some standard computations, we have that the lateral inner flow across $Q_{\ell^\pm_{1,\gamma}\cdot\ell}$ is given by

$$\int_{t\in(t^-_\gamma,t^+_\gamma)} \big| \mathsf{Tr}^{\mathrm{in}}\big((1,\mathbf{b}), Q_{\ell^\pm_{1,\gamma},\ell}(t)\big)\cdot \mathbf{n}(t)\big|\, \mathcal{H}^d \llcorner_{\partial Q_{\ell^\pm_{1,\gamma},\ell}} = I_{2,\gamma} + I^+_{1,\gamma} + I^-_{1,\gamma},$$

where

$$I_{2,\gamma} := \int_{t^-_\gamma}^{t^+_\gamma} \int_{L_{2,\gamma}(t)} \big|\big(\dot\gamma(t) - \mathbf{b}^-\big)\cdot \mathbf{e}^\perp\big|\, \mathcal{H}^{d-1} dt \tag{11.6}$$

and

$$I^+_{1,\gamma} := \int_{t^-_\gamma}^{t^+_\gamma} \int_{L^+_{1,\gamma}(t)} \big|\big(\partial_t \ell^+_{1,\gamma}\mathbf{e}_1 + \dot\gamma - \mathbf{b}^-\big)\cdot \mathbf{n}\big|\, \mathcal{H}^{d-1} dt, \tag{11.7a}$$

$$I^-_{1,\gamma} := \int_{t^-_\gamma}^{t^+_\gamma} \int_{L^-_{1,\gamma}(t)} \big|\big(- \partial_t \ell^-_{1,\gamma}\mathbf{e}_1 - \dot\gamma + \mathbf{b}^-\big)\cdot \mathbf{n}\big|\, \mathcal{H}^{d-1} dt. \tag{11.7b}$$

To simplify notations we put an apex $-$ to denote the inner trace of $\mathbf{b}$ on the boundary of a Lipschitz set, and we recall that $\mathbf{n} = (1, -\nabla_{y^\perp}\ell^\pm_{1,\gamma}(t))/|(1, -\nabla_{y^\perp}\ell^\pm_{1,\gamma}(t))|$.

11.2.2. *Estimates on the flux.* The following lemmas will be proved in the next section.

Lemma 11.3 (Transversal flux)**.** *For all $(\bar t, \bar x) \in K^\tau_{\delta_c,\bar r}$, $r < \bar r$ it holds*

$$\int \frac{1}{\mathcal{L}^d(\bar Q)} I_{2,\gamma}\, \eta(d\gamma) \le C_{d-1}\tau |D\mathbf{b}|\big(B^{d+1}_{r+2\ell}(\bar t, \bar x)\big).$$

Lemma 11.4 (Non-transversal flux)**.** *For all $(\bar t, \bar x) \in K^\tau_{\delta_c,\bar r}$, $r < \bar r$ it holds*

$$\int \frac{1}{\mathcal{L}^d(\bar Q)} I^\pm_{1,\gamma}\, \eta(d\gamma) \le C_{d-1}\tau |D\mathbf{b}|\big(B^{d+1}_{r+2\ell}(\bar t, \bar x)\big).$$

From these results we deduce the following proposition.

Proposition 11.5. *For every point $(\bar t,\bar x) \in K^\tau_{\delta_c,\bar r}$ and $\varepsilon > 0$, there exists a family of $(1,\mathbf{b})$-proper balls $\{B^{d+1}_r(\bar t,\bar x)\}_r$, with $r < \bar r$ having 0 as a Lebesgue point, such that Assumption 7.6 holds with constant*

$$\varpi_r(\bar t,\bar x) \leq C_{d-1}\tau|D\mathbf{b}|\big(B^{d+1}_{r+\varepsilon}(\bar t,\bar x)\big).$$

Proof. First of all, by the regularity assumptions on $\mathbf{b}$, it follows that the lateral boundary of $Q_{\ell^\pm_{1,\gamma},\ell}$ is inner regular, so that Point (1) of Assumption 7.6 is verified. Moreover by construction Point (2) holds with constant $M = \delta_1$, being $\delta_1\ell \leq \ell^\pm_{1,\gamma}$. Finally for $2\ell < \varepsilon$ one applies the above Lemmas to recover Point (3). □

By the above proposition and Proposition 11.1 we thus conclude that

Theorem 11.6. *Assumptions 8.12 holds for a vector field $\mathbf{b} \in L^1_{\mathrm{loc}}(\mathbb{R},\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^d))$.*

Proof. By choosing the local balls accordingly to Proposition 11.5 on $K^\tau_{\delta_c,\bar r}$ and according to Proposition 11.1 in the remaining points, one sees that the measure ϖ^τ can be taken to be

$$\varpi^\tau = C_{d-1}\tau|D\mathbf{b}|\llcorner_{K^{\mathrm{a.c.}}\cup K^\tau_{\delta_c,\bar r}} + C_d|D\mathbf{b}|\llcorner_{\mathbb{R}^{d+1}\setminus(K^{\mathrm{a.c.}}\cup K^\tau_{\delta_c,\bar r})},$$

where $K^{\mathrm{a.c.}}$ is a compact set made of Lebesgue points for $D^{\mathrm{a.c.}}\mathbf{b}$. In particular the measure ϖ^τ can be made arbitrarily small by letting first $\tau \to 0$ and then $\bar r \to 0$, so to have $K^{\mathrm{a.c.}} \cup K^\tau_{\delta_c,\bar r} \nearrow \mathbb{R}^{d+1}$. □

12. Flux estimates and proof of Lemmata 11.3 and 11.4

Here we prove the two lemmata that allow to control the boundary flux of $(1,\mathbf{b})$ on $Q_{\ell^\pm_{1,\gamma},\ell}$. We will just prove the case $\overline{\mathtt{m}}_1 < -\tau$, being the second case completely analogous by inverting time and the case $\ell^\pm_{1,\gamma} = \bar\ell_1$ a simple variation of the first situation.

Observe that for a given positive Borel function $f(x,y)$ it holds

$$\begin{aligned}\int_\Gamma\int_{L_{2,\gamma}(t)} f(\gamma(t),y)\,\mathcal{H}^{d-1}(dy)\eta(d\gamma) &\leq \int_\Gamma\int_{\bar L_2} f(\gamma(t),y)\,\mathcal{H}^{d-1}(dy)\eta(d\gamma) \\ &= \int_{(B^{d+1}_{\bar r}(\bar t,\bar x))_t}\int_{\bar L_2} f(x,y)\,\mathcal{H}^{d-1}(dy)\mathcal{L}^d(dx),\end{aligned} \tag{12.1}$$

where we used the notation

$$\bar L_2 := \big\{(y_1,y^\perp), |y_1| \leq \bar\ell_1, |y^\perp| = \ell\big\},$$

and $(B^{d+1}_{\bar r}(\bar t,\bar x))_t$ is the t-time section of the ball where $(\mathtt{p}_{t,x})_\sharp\eta$ is concentrated.

12.1. **Proof of Lemma 11.3.** We recall that the quantity $I_{2,\gamma}$ was defined in (11.6) as

$$I_{2,\gamma} = \int_{t^-_\gamma}^{t^+_\gamma}\int_{L_{2,\gamma}(t)}\big|\big(\dot\gamma(t) - \mathbf{b}(t,\gamma(t)+y-)\big)\cdot\mathbf{e}^\perp\big|\,\mathcal{H}^{d-1}(dy)dt.$$

Since this quantity is defined for a curve γ and then integrated in γ, by the a.c. of the projection of η on $\{t\}\times\mathbb{R}^d$ we will consider $\mathbf{b}$ defined on suitable planes passing through $\gamma(t)$. We will also avoid putting the $-$ sign to remember that we are taking the inner trace: for this term indeed, begin the surface $L_{2,\gamma}$ a subset of $\gamma + \{|y_1| < \bar\ell_1\} \times B^{d-1}_\ell(0)$, one can assume that it is made of Lebesgue points.

Proof of Lemma 11.3. Observe first that, for fixed t, adding and subtracting the term $\mathbf{b}(t,(\gamma_1(t) + y_1,\gamma^\perp(t)))$ and using the triangular inequality, we can write

$$\begin{aligned}&\int_{L_{2,\gamma}(t)}\Big|\big(\dot\gamma(t) - \mathbf{b}(t,(\gamma_1(t)+y_1,\gamma^\perp(t)+y^\perp))\big)\cdot\mathbf{e}^\perp\Big|\,\mathcal{H}^{d-1}(dy) \\ &\qquad\leq \int_{L_{2,\gamma}(t)}\Big|\Big[\mathbf{b}\big(t,\big(\gamma_1(t)+y_1,\gamma^\perp(t)+y^\perp\big)\big) - \mathbf{b}\big(t,\big(\gamma_1(t)+y_1,\gamma^\perp(t)\big)\big)\Big]\cdot\mathbf{e}^\perp\Big|\,\mathcal{H}^{d-1}(dy) \\ &\qquad\quad+ \int_{L_{2,\gamma}(t)}\Big|\Big[\mathbf{b}\big(t,\big(\gamma_1(t)+y_1,\gamma^\perp(t)\big)\big) - \dot\gamma(t)\Big]\cdot\mathbf{e}^\perp\Big|\,\mathcal{H}^{d-1}(dy).\end{aligned} \tag{12.2}$$

Integrating (12.2) in η and dividing by $\mathcal{L}^d(\bar{Q}) = 2\omega_{d-1}\ell^{d-1}\bar{\ell}_1$, we have that

$$\begin{aligned}
&\frac{1}{\mathcal{L}^d(\bar{Q})}\int_\Gamma \int_{L_{2,\gamma(t)}} \big|\big(\dot{\gamma}(t) - \mathbf{b}(t,(\gamma_1(t)+y_1,\gamma^\perp(t)+y^\perp))\big)\cdot \mathbf{e}^\perp\big|\,\mathcal{H}^{d-1}(dy)\eta(d\gamma) \\
&\quad\le \frac{1}{\mathcal{L}^d(\bar{Q})}\int_\Gamma \int_{L_{2,\gamma(t)}} \Big|\mathbf{b}^\perp\big(t,(\gamma_1(t)+y_1,\gamma^\perp(t)+y^\perp)\big) - \mathbf{b}^\perp\big(t,(\gamma_1(t)+y_1,\gamma^\perp(t))\big)\Big|\,\mathcal{H}^{d-1}(dy)\eta(d\gamma) \\
&\qquad + \frac{1}{\mathcal{L}^d(\bar{Q})}\int_\Gamma \int_{L_{2,\gamma(t)}} \Big|\mathbf{b}^\perp\big(t,(\gamma_1(t)+y_1,\gamma^\perp(t))\big) - (\dot{\gamma}(t))^\perp\Big|\,\mathcal{H}^{d-1}(dy)\eta(d\gamma) \\
&\quad =: S_2^{\mathrm{BV}}(t) + S_2^{\mathrm{av}}(t).
\end{aligned}$$

We now proceed to estimate the two terms separately.

Step 1: estimate of the term $S_2^{\mathrm{BV}}(t)$. By (12.1) we have

$$\begin{aligned}
S_2^{\mathrm{BV}}(t) &= \frac{1}{\mathcal{L}^d(\bar{Q})}\int_\Gamma \int_{L_{2,\gamma(t)}} \Big|\mathbf{b}^\perp\big(t,(\gamma_1(t)+y_1,\gamma^\perp(t)+y^\perp)\big) - \mathbf{b}^\perp\big(t,(\gamma_1(t)+y_1,\gamma^\perp(t))\big)\Big|\,\mathcal{H}^{d-1}(dy)\eta(d\gamma) \\
&\le \frac{1}{\mathcal{L}^d(\bar{Q})}\int_{(B^{d+1}_{\bar{r}}(\bar{t},\bar{x}))_t} \int_{\bar{L}_2} \Big|\mathbf{b}^\perp\big(t,(x_1+y_1,x^\perp+y^\perp)\big) - \mathbf{b}^\perp\big(t,(x_1+y_1,x^\perp)\big)\Big|\,\mathcal{H}^{d-1}(dy)\mathcal{L}^d(dx).
\end{aligned}$$

By Fubini and the one dimensional slicing of BV functions [Zie89, Theorem 5.3.5], we deduce

$$\begin{aligned}
S_2^{\mathrm{BV}}(t) &\le \frac{1}{\mathcal{L}^d(\bar{Q})}\int_{\bar{L}_2}\bigg[\int_{(B^{d+1}_{\bar{r}}(\bar{t},\bar{x}))_t} \big|D^\perp \mathbf{b}^\perp_t\big|\big(x_1+y_1,(x^\perp,x^\perp+y^\perp)\big)\,\mathcal{L}^d(dx)\bigg]\mathcal{H}^d(dy) \\
&\le \frac{1}{\mathcal{L}^d(\bar{Q})}\int_{\bar{L}_2} \ell\big|D^\perp \mathbf{b}^\perp_t\big|\big((B^{d+1}_{\bar{r}+\ell+\bar{\ell}_1}(\bar{t},\bar{x}))_t\big)\,\mathcal{H}^d(dy) \\
&\le \frac{1}{2\omega_{d-1}\ell^{d-1}\bar{\ell}_1}\cdot 2(d-1)\omega_{d-1}\ell^{d-2}\bar{\ell}_1\cdot \ell|D^\perp \mathbf{b}^\perp_t|\big((B^{d+1}_{\bar{r}+(1+\tau)\ell}(\bar{t},\bar{x}))_t\big) \\
&\le C_{d-1}|D^\perp \mathbf{b}^\perp_t|\big(B^{d+1}_{r+2\ell}(\bar{t},\bar{x}))_t\big).
\end{aligned}$$

Finally, integrating in time and using (11.2), we obtain

$$\int S_2^{\mathrm{BV}}(t)\,\mathcal{L}^1(dt) \le C_{d-1}|D^\perp \mathbf{b}^\perp|\big(B^{d+1}_{\bar{r}+2\ell}(\bar{t},\bar{x})\big) \le C_{d-1}\tau|D\mathbf{b}|\big(B^{d+1}_{\bar{r}+2\ell}(\bar{t},\bar{x})\big). \tag{12.3}$$

Step 2 Estimate of the term $S_2^{\mathrm{av}}(t)$. We have using again (12.1)

$$\begin{aligned}
S_2^{\mathrm{av}}(t) &= \frac{1}{\mathcal{L}^d(\bar{Q})}\int_\Gamma \int_{L_{2,\gamma(t)}} \Big|\mathbf{b}^\perp\big(t,\big(\gamma_1(t)+y_1,\gamma^\perp(t)\big)\big) - (\dot{\gamma}(t))^\perp\Big|\,\mathcal{H}^{d-1}(dy)\eta(d\gamma) \\
&\le \frac{1}{\mathcal{L}^d(\bar{Q})}\int_{(B^{d+1}_{r}(\bar{t},\bar{x}))_t} \int_{\bar{L}_2} \Big|\mathbf{b}^\perp\big(t,\big(x_1+y_1,x^\perp\big)\big) - \mathbf{b}^\perp(t,x)\Big|\,\mathcal{H}^{d-1}(dy)\mathcal{L}^d(dx),
\end{aligned}$$

and arguing as before, using Fubini and the one dimensional slicing of BV functions, we obtain

$$\begin{aligned}
S_2^{\mathrm{av}}(t) &\le \frac{1}{\mathcal{L}^d(\bar{Q})}\int_{\bar{L}_2}\int_{(B^{d+1}_{r}(\bar{t},\bar{x}))_t} \Big|\mathbf{b}^\perp\big(t,\big(x_1+y_1,x^\perp\big)\big) - \mathbf{b}^\perp(t,x)\Big]\cdot e^\perp\Big|\,\mathcal{L}^d(dx)\mathcal{H}^{d-1}(dy) \\
&\le \frac{1}{\mathcal{L}^d(\bar{Q})}\int_{\bar{L}_2} \bar{\ell}_1|D_1\mathbf{b}_t|\big((B^{d+1}_{\bar{r}+\bar{\ell}_1}(\bar{t},\bar{x}))_t\big)\,\mathcal{H}^{d-1}(dy) \\
&\le \frac{1}{2\omega_{d-1}\ell^{d-1}\bar{\ell}_1}\cdot 2(d-1)\omega_{d-1}\ell^{d-2}\bar{\ell}_1\cdot \bar{\ell}_1|D_1\mathbf{b}_t|\big((B^{d+1}_{r+\bar{\ell}_1}(\bar{t},\bar{x}))_t\big) \\
&\le C_{d-1}\frac{\bar{\ell}_1}{\ell}|D_1\mathbf{b}_t|\big((B^{d+1}_{r+\bar{\ell}_1}(\bar{t},\bar{x}))_t\big) \\
&\le C_{d-1}\tau|D_1\mathbf{b}_t|\big((B^{d+1}_{r+\bar{\ell}_1}(\bar{t},\bar{x}))_t\big).
\end{aligned}$$

Integrating in time we obtain

$$\int S_2^{\mathrm{av}}(t)\,\mathcal{L}^1(dt) \le C_{d-1}\tau|D\mathbf{b}|\big(B^{d+1}_{r+\bar{\ell}_1}(\bar{t},\bar{x})\big). \tag{12.4}$$

Summing up (12.3) and (12.4) we finally deduce, for $\tau \ll 1$,

$$
\begin{aligned}
\frac{1}{\mathcal{L}^d(\bar{Q})} \int_\Gamma I_{2,\gamma}\, \eta(d\gamma) &= \frac{1}{\mathcal{L}^d(\bar{Q})} \int_\Gamma \int_{t_\gamma^-}^{t_\gamma^+} \int_{L_{2,\gamma}(t)} \big| \big(\dot{\gamma}(t) - \mathbf{b}(t, x+y)\big) \cdot \mathbf{e}^\perp \big| \, \mathcal{H}^{d-1}(dy) \mathcal{L}^1(dt) \eta(d\gamma) \\
&\le \int S_2^{\mathrm{BV}}(t)\, \mathcal{L}^1(dt) + \int S_2^{\mathrm{av}}(t)\, \mathcal{L}^1(dt) \\
&\le C_{d-1} \tau |D\mathbf{b}| \big(B^{d+1}_{r+2\ell}(\bar{t}, \bar{x})\big).
\end{aligned}
$$

which is the claim. $\square$

12.2. **Proof of Lemma 11.4.** The proof of Lemma 11.4 depends heavily on the shape of the cylinders, which cancel the effect of the divergence thanks to the choice of $\ell^\pm_{1,\gamma}(t, y^\perp)$. The goal is to show Lemma 11.4, i.e.

$$
\frac{1}{\mathcal{L}^d(\bar{Q})} \int I^\pm_{1,\gamma}\, \eta(d\gamma) \le C_{d-1} \tau |D\mathbf{b}| \big(B_{r+2\ell}\big).
$$

We will prove only the estimate for $I^{1,+}_\gamma$ being the other case identical.

Proof of Lemma 11.4 for $I^+_{1,\gamma}$. Recall that the quantity $I^+_{1,\gamma}$ was defined in (11.7) as

$$
\begin{aligned}
I^+_{1,\gamma} &= \int_{\bar{t}-\bar{r}/2}^{\bar{t}+\bar{r}/2} \int_{L^+_{1,\gamma}(t)} \big| \big(\dot{\ell}^+_{1,\gamma} \mathbf{e}_1 + \dot{\gamma} - \mathbf{b}^-\big) \cdot \mathbf{n} \big| \, \mathcal{H}^{d-1}(dy) dt \\
&\le \int_{\bar{t}-\bar{r}/2}^{\bar{t}+\bar{r}/2} \int_{|y^\perp|<\ell} \big| \dot{\ell}^+_{1,\gamma} + \dot{\gamma}_1 - \mathbf{b}_1\big(t, \gamma(t) + (\ell^+_{1,\gamma}(t), y^\perp)\big) \big| \, \mathcal{L}^{d-1}(dy^\perp) dt \\
&\quad + \int_{\bar{t}-\bar{r}/2}^{\bar{t}+\bar{r}/2} \int_{|y^\perp|<\ell} \big| \big(\dot{\gamma}^\perp - \mathbf{b}^\perp\big(t, \gamma(t) + (\ell^+_{1,\gamma}(t), y^\perp)\big)\big) \cdot \nabla_{y^\perp} \ell^+_{1,\gamma}(t, y^\perp) \big| \, \mathcal{L}^{d-1}(dy^\perp) dt \\
&\le \int_{\bar{t}-\bar{r}/2}^{\bar{t}+\bar{r}/2} \int_{|y^\perp|<\ell} \Big| \mathbf{b}_1\big(t, \gamma(t) + (\ell^+_{1,\gamma}(t), y^\perp)\big) - \mathbf{b}_1\big(t, \gamma(t) + ((\delta_1 + \delta_c)\ell, 0)\big) - \dot{\ell}^+_1(t) \Big| \, \mathcal{L}^{d-1}(dy^\perp) dt \\
&\quad + \int_{\bar{t}-\bar{r}/2}^{\bar{t}+\bar{r}/2} \int_{|y^\perp|<\ell} \big| \mathbf{b}_1\big(t, \gamma(t) + ((\delta_1 + \delta_c)\ell, 0)\big) - \dot{\gamma}_1(t) \big| \, \mathcal{L}^{d-1}(dy^\perp) dt \\
&\quad + \int_{\bar{t}-\bar{r}/2}^{\bar{t}+\bar{r}/2} \int_{|y^\perp|<\ell} \big| \big(\dot{\gamma}^\perp - \mathbf{b}^\perp\big(t, \gamma(t) + (\ell^+_{1,\gamma}(t), y^\perp)\big)\big) \cdot \nabla_{y^\perp} \ell^+_{1,\gamma}(t, y^\perp) \big| \, \mathcal{L}^{d-1}(dy^\perp) dt.
\end{aligned}
$$

Integrating at a fixed time t the above equation in γ and dividing by the area of $\bar{Q}$, we have that

$$
\begin{aligned}
&\frac{1}{\mathcal{L}^d(\bar{Q})} \int_\Gamma \int_{|y^\perp|<\ell} \big| \big(\dot{\ell}^+_{1,\gamma} \mathbf{e}_1 + \dot{\gamma} - \mathbf{b}^-\big) \cdot \mathbf{n} \big| \, \mathcal{L}^{d-1}(dy^\perp) \eta(d\gamma) \\
&\le \frac{1}{\mathcal{L}^d(\bar{Q}) \delta_1 \ell} \int_\Gamma \int_{|y^\perp|<\ell} \Big| \mathbf{b}_1\big(t, \gamma(t) + (\ell^+_{1,\gamma}(t, y^\perp), y^\perp)\big) - \mathbf{b}_1\big(t, \gamma(t) + ((\delta_1 + \delta_c)\ell, 0)\big) - \dot{\ell}^+_1(t) \Big| \, \mathcal{L}^{d-1}(dy^\perp) \eta(d\gamma) \\
&\quad + \frac{1}{\mathcal{L}^d(\bar{Q}) \delta_1 \ell} \int_\Gamma \int_{|y^\perp|<\ell} \Big| \mathbf{b}_1\big(t, \gamma(t) + ((\delta_1 + \delta_c)\ell, 0)\big) - \dot{\gamma}_1(t) \Big| \, \mathcal{L}^{d-1}(dy^\perp) \eta(d\gamma) \\
&\quad + \frac{1}{\mathcal{L}^d(\bar{Q}) \delta_1 \ell} \int_\Gamma \int_{|y^\perp|<\ell} \Big| \big(\mathbf{b}^\perp\big(t, \gamma(t)\big) - \mathbf{b}^\perp\big(t, \gamma(t) + (\ell^+_{1,\gamma}(t), y^\perp)\big)\big) \cdot \nabla_{y^\perp} \ell^+_{1,\gamma}(t, y^\perp) \Big| \, \mathcal{L}^{d-1}(dy^\perp) \eta(d\gamma) \\
&=: S_1^{\mathrm{RL}}(t) + S_1^{\mathrm{av}}(t) + S_1^{\mathrm{tr}}(t).
\end{aligned}
$$

We now proceed to estimate the terms separately.

<u>*Step 1*</u>*: estimate of the term* $S_1^{\mathrm{av}}(t)$. We have

$$\begin{aligned}
S_2^{\mathrm{av}}(t) &= \frac{1}{\mathcal{L}^d(\bar{Q})}\int_\Gamma \int_{|y^\perp|\le \ell} \Big|\mathbf{b}_1\big(t,\gamma(t)+((\delta_1+\delta_c)\ell,0)\big)-\dot{\gamma}_1(t)\Big|\,\mathcal{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&= \frac{1}{2\bar{\ell}_1}\int_{(B^{d+1}_r(\bar{t},\bar{x}))_t}\int_{|y^\perp|\le\ell}\Big|\mathbf{b}_1\big(t,x+((\delta_1+\delta_c)\ell,0)\big)-\mathbf{b}_1(t,x)\Big|\,\mathcal{L}^d(dx)\\
&\le \frac{(\delta_1+\delta_c)\ell}{2\bar{\ell}_1}|D\mathbf{b}_t|\big((B^{d+1}_{\bar{r}+\ell}(\bar{t},\bar{x}))_t\big)\\
&\le \tau|D\mathbf{b}_t|\big((B^{d+1}_{\bar{r}+\ell}(\bar{t},\bar{x}))_t\big)
\end{aligned}$$

by Fubini and the one dimensional slicing of BV.

<u>*Step 2*</u>*: estimate of the term* $S_1^{\mathrm{RL}}(t)$. By the definition of $\ell^+_{1,\gamma}(t,y^\perp)$ through the ODE (11.5b) we obtain

$$\begin{aligned}
S_1^{\mathrm{RL}}(t) &= \frac{1}{\mathcal{L}^d(\bar{Q})}\int_\Gamma\int_{|y^\perp|<\ell}\Big|\mathbf{b}_1\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t,y^\perp),y^\perp)\big)-\mathbf{b}_1\big(t,\gamma(t)+((\delta_1+\delta_c)\ell,0)\big)-\dot{\ell}^+_1(t,y^\perp)\Big|\,\mathcal{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&= \frac{1}{\mathcal{L}^d(\bar{Q})}\int_\Gamma\int_{|y^\perp|<\ell}\Big|(\mathbf{b}_1-\mathcal{U}_1)\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t,y^\perp),y^\perp)\big)-(\mathbf{b}_1-\mathcal{U}_1)\big(t,\gamma(t)+((\delta_1+\delta_c)\ell,0)\big)\Big|\,\mathcal{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&\le \frac{1}{\mathcal{L}^d(\bar{Q})}\int_\Gamma\int_{|y^\perp|<\ell}\Big|(\mathbf{b}_1-\mathcal{U}_1)\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t,y^\perp),y^\perp)\big)-(\mathbf{b}_1-\mathcal{U}_1)\big(t,\gamma(t)+((\delta_1+\delta_c)\ell,y^\perp)\big)\Big|\,\mathcal{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&\quad+\frac{1}{\mathcal{L}^d(\bar{Q})}\int_\Gamma\int_{|y^\perp|<\ell}\Big|(\mathbf{b}_1-\mathcal{U}_1)\big(t,\gamma(t)+(\delta_1+\delta_c,y^\perp)\big)-(\mathbf{b}_1-\mathcal{U}_1)\big(t,\gamma(t)+((\delta_1+\delta_c)\ell,0)\big)\Big|\,\mathcal{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&\le \frac{1}{\mathcal{L}^d(\bar{Q})}\int_\Gamma |D\mathbf{b}-D\mathcal{U}|\big(\gamma(t)+\big\{|y^\perp|<\ell,\delta_1\ell<y_1<\bar{\ell}_1\big\}\big)\eta(d\gamma)\\
&\quad+\frac{1}{\mathcal{L}^d(\bar{Q})}\int_{(B^{d+1}_r(\bar{t},\bar{x}))_t}\int_{|y^\perp|<\ell}\Big|(\mathbf{b}_1-\mathcal{U}_1)\big(t,x+(\delta_1+\delta_c,y^\perp)\big)-(\mathbf{b}_1-\mathcal{U}_1)\big(t,x+((\delta_1+\delta_c)\ell,0)\big)\Big|\,\mathcal{L}^{d-1}(dy^\perp)\mathcal{L}^d(dx)\\
&\le \frac{1}{2\omega_{d-1}\ell^{d-1}\bar{\ell}_1}\cdot 2\omega_{d-1}\ell^{d-1}\bar{\ell}_1\cdot|D\mathbf{b}_t-D\mathcal{U}|\big((B^{d+1}_{\bar{r}+2\ell}(\bar{t},\bar{x}))_t\big)\\
&\quad+\frac{1}{2\omega_{d-1}\ell^{d-1}\bar{\ell}_1}\cdot 2\omega_{d-1}\ell^{d-1}\ell\cdot|D^\perp\mathbf{b}_t-D^\perp\mathcal{U}|\big((B^{d+1}_{\bar{r}+2\ell}(\bar{t},\bar{x}))_t\big)\\
&\le C_{d-1}\left(\tau+\frac{\ell\tau^2}{\bar{\ell}_1}\right)|D\mathbf{b}_t|\big((B^{d+1}_{\bar{r}+2\ell}(\bar{t},\bar{x}))_t\big)\le C_{d-1}\tau|D\mathbf{b}_t|\big((B^{d+1}_{\bar{r}+2\ell}(\bar{t},\bar{x}))_t\big),
\end{aligned}$$

where we applied (11.3) and (11.2) to control the normal derivative.

<u>*Step 3*</u>*: estimate of the term* $S_1^{\mathrm{tr}}(t)$. For the last term, recalling that $y^\perp\mapsto\ell^+_{1,\gamma}$ is δ_c-Lipschitz by Lemma 11.2, we have

$$\begin{aligned}
S_1^{\mathrm{tr}}(t) &= \frac{1}{\mathcal{L}^d(\bar{Q})}\int_\Gamma\int_{|y^\perp|\le\ell}\big|\big(\dot{\gamma}^\perp-\mathbf{b}^\perp\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t,y^\perp),y^\perp)\big)\big)\cdot\nabla_{y^\perp}\ell^+_{1,\gamma}(t,y^\perp)\big|\,\mathcal{L}^{d-1}(dy^\perp)\eta(d\gamma)\\
&\le \frac{\delta_c}{\mathcal{L}^d(\bar{Q})}\int_\Gamma\int_{|y^\perp|\le\ell}\Big|\mathbf{b}^\perp\big(t,\gamma(t)+(\ell^+_{1,\gamma}(t,y^\perp),y^\perp)\big)-\mathbf{b}^\perp\big(t,\gamma(t)\big)\Big|\,\mathcal{L}^{d-1}(dy^\perp)\eta(d\gamma).
\end{aligned}$$

Again enlarging the set $Q_{\ell^\pm_{1,\gamma},\ell}(t)$ to $\bar{Q}$ we obtain

$$\begin{aligned}
S_1^{\mathrm{tr}}(t) &\le \frac{\delta_c}{\mathcal{L}^d(\bar{Q})}\int_{|y^\perp|\le\ell}\big(\ell|D^\perp\mathbf{b}|+\bar{\ell}_1|D_1\mathbf{b}|\big)\big((B^{d+1}_{r+\ell+\ell_1}(\bar{t},\bar{x}))_t\big)\,\mathcal{L}^{d-1}(dy^\perp)\\
&\le \frac{\delta_c}{2\omega_{d-1}\ell^{d-1}\bar{\ell}_1}\cdot\omega_{d-1}\ell^{d-1}\cdot(\tau+1)\ell|D\mathbf{b}|\big((B^{d+1}_{\bar{r}+2\ell}(\bar{t},\bar{x}))_t\big)\\
&\le C_{d-1}\tau|D\mathbf{b}|\big((B^{d+1}_{\bar{r}+2\ell}(\bar{t},\bar{x}))_t\big),
\end{aligned}$$

by the choice of $\delta_c\le\tau^2$.

Integrating in time and summing up the three terms we conclude the proof of Lemma 11.4. $\square$

Glossary

$C(X,Y)$: space of continuous functions over X. 16

A: generic set. 15
$A \Subset B$: set A whose compact closure is contained in B. 15
A_x: x section of $A \subset X \times Y$. 15
$\mathbf{Adm}(\mu_i)$: sets of admissible transference plans. 19
A^ℓ_γ: set of intersecting curves. 47
$A^\pm$: subset of $\partial\Omega$ where trajectories are exiting or entering, respectively. 42
$⨍_A f\,\mu$: average integral on the sets A. 17
$A(x)$: x section of $A \subset X \times Y$. 15

B: generic vector field in $\mathbb{R}^{d+1}$. 15
$\mathbf{b} = (b_i)_{i=1}^d$: vector. 15
$B^d_r(x)$: balls of radius r centered at $x \in \mathbb{R}^d$. 15
$\mathcal{O}(f)$: notation for constant of the order f. 17
$\partial\Omega$: frontier of a set in $\mathbb{R}^d$. 15
$\mathbf{b}_t$: equivalent to $\mathbf{b}(t)$ for time dependent vector fields. 15
$\mathrm{BV}_{\mathrm{loc}}$: space of locally BV functions. 16

C: generic constant. 17
C_d: dimensional constant. 17
$\mathbb{1}_A$: characteristic function of the set A. 15
$C^k(\mathbb{R}^d, \mathbb{R}^{d'})$: space of functions on $\mathbb{R}^d$ with continuous derivatives up to order k. 16
$\mathrm{clos}(A,B)$: relative closure of the set A in B. 15
$\mathrm{clos}\,A$: closure of the set A. 15
$*$: convolution in $\mathbb{R}^d$. 17
$\mathrm{Cyl}^{r,L}_{t,x}$: $\rho(1,\mathbf{b})$-proper cylinder. 24

$D_{\mathbf{e}} f$: directional derivative of f along $\mathbf{e}$. 16
$D^{\mathrm{a.c.}}\mathbf{b}$: absolutely continuous part of Df. 19
$D^{\mathrm{cantor}}\mathbf{b}$: Cantor part of $D^{\mathrm{sing}} f$. 19
δ_x: Dirac mass at x. 16
δ_1: maximal shrinking coefficient of an approximate cylinder of flow. 72
Df: differential of the function f. 15
$\mu = \int \mu_\alpha\, f_\sharp\mu(d\alpha)$: disintegration of μ w.r.t. the partition $\{A_\alpha\}_\alpha$. 16
$\mathrm{div}\,\mathbf{b}$: divergence of the vector field $\mathbf{b}$. 15
$D^{\mathrm{jump}}\mathbf{b}$: jump part of $D^{\mathrm{sing}} f$. 19
$\mathcal{D}(f)$: domain of the function f. 16
$D^{\mathrm{sing}}\mathbf{b}$: singular part of Df. 19

$\mathbf{e}$: unit vector. 16
$E\mathbf{b}$: symmetric part of the derivative $D\mathbf{b}$. 19
$E_{\mathfrak{a}}$: equivalence classes of $\sim$. 60
E^ℓ_γ: set of curves not contained in $\mathrm{supp}\,\phi^\ell_\gamma$. 45
E^f_h: upper level set of the function f. 20
E_h: upper level set of a function. 20
$\bar\ell_1$: starting shape of the approximate cylinder of flow in the BV case. 72
$\ell^\pm_{1,\gamma}$: evolution of the $\mathbf{e}_1$-boundary of the approximate cylinder. 72
η: Lagrangian representation. 17
η^{cr}: restriction fo η to $\varGamma^{\mathrm{Cr}}$. 51
η^i_Ω: push forward of η by $\mathtt{R}^i_\Omega$. 33
η^Ξ: restriction of η to $\varXi$. 53

f: generic function. 15

ϕ**:** particular functions used in the paper, usually with an index/apex. 17
$\phi^{\delta,\pm}$**:** inner/outer distance functions from a set. 22
$\Phi^{\ell}_{\mathrm{enter}}(\gamma)$**:** functional computing intersecting curves across ϕ^{ℓ}_{γ}. 47
$\Phi^{\ell}_{\mathrm{exit}}(\gamma)$**:** functional computing the curves exiting the cylinder ϕ^{ℓ}_{γ}. 45
ϕ^{ℓ}_{γ}**:** approximate cylinder of flow. 44
$\mathfrak{f}^{\mathrm{in}}(\Omega)$**:** untangling functional for η^{in}. 55
$\mathfrak{f}^{\mathrm{out}}(\Omega)$**:** untangling functional for η^{out}. 55
$\mathfrak{A}$**:** suitable set of indexes. 60
$f\llcorner_A$**:** restriction of the function f to the set A. 16
f^r_x**:** rescaled f about $x \in \mathbb{R}^d$. 16
$f_\sharp\mu$**:** push-forward of the measure μ through f. 16
$f(\bar{x}\pm)$**:** right left limit of a 1d function at $\bar{x}$. 17

γ**:** curve define in an interval of time. 17
$\gamma \sim \gamma'$**:** equivalent relation among untangled trajectories. 60
$g \circ f$**:** composition of two functions. 16
$\mathrm{Graph}\, f$**:** graph of the function f. 16
$\mathrm{Graph}\, \gamma$**:** Graph of the a.c. curve γ in the closed interval of definition. 18

$\hat{\mathtt{f}}$**:** quotient map for $\{E_{\mathfrak{a}}\}_{\mathfrak{a}}$. 61
$\mathcal{H}^d$**:** d-dimensional Hausdorff measure. 16

$I_{2,\gamma}$**:** flow across $L_{2,\gamma}$. 73
id**:** identity function. 17
$I_\gamma = (t^-_\gamma, t^+_\gamma)$**:** interval of definition of the curve γ. 17
$I^-_{1,\gamma}$**:** flow across $L^-_{1,\gamma}$. 75
$\mathrm{int}\, A$**:** interior of the set A. 15
$\int f\, dx$**:** integral of a Borel function f w.r.t. $\mathcal{L}^d$. 16
$\int f\, \mu$**:** integral of a Borel function f w.r.t. μ. 16
$I^+_{1,\gamma}$**:** flow across $L^+_{1,\gamma}$. 75

J**:** jump set of a BV function. 19

$K^{\varepsilon,\varepsilon'}_{\bar{r}} \subset K^{\varepsilon}$**:** compact subset of $\partial\Omega$ defined in Lemma 4.14. 28
K^n**:** projection of $\mathcal{K}^n$. 58
$K^{\tau}_{\delta_c,\bar{r}}$**:** compact set with suitable local covering. 71
$K^{\tau,\pm}$**:** compact sets where the untangling functionals are controlled. 57, 59

L**:** scale constant. 17
$L^1(\mu, Y)$**:** space of functions whose modulus is μ-integrable. 16
$\bar{L}_2$**:** lateral boundary of $\bar{Q}$ with normal $\mathbf{e}^{\perp}_1$. 76
$L_{2,\gamma}$**:** lateral boundary of $Q_{\ell^{\pm}_{1,\gamma},\ell}$ with normal $\mathbf{e}^{\perp}_1$. 73
$\langle f, \psi\rangle$**:** distribution f evaluated on ψ. 15
$\mathcal{L}^d$**:** Lebesgue measure in $\mathbb{R}^d$. 16
$L^\infty(\mu, Y)$**:** space of functions with μ-essentially bounded Y-norm. 16
$L^{\pm}_{1,\gamma}$**:** lateral boundary of $Q_{\ell^{\pm}_{1,\gamma},\ell}$ given by the graph of $\ell^{\pm}_{1,\gamma}$. 73

$\mathcal{M}(X)$**:** set of Radon measures. 16
m**:** image measure $\hat{\mathtt{f}}_\sharp\eta$. 61
m**:** direction of the variation in the rank-one property. 20
$\mathcal{K}^n$**:** compact subset of Γ of trajectories with existence interval $\geq 2^{1-n}$. 58
$\mathcal{S}$**:** sets of curves with the same initial point. 53
$\mathcal{M}_b(X)$**:** set of bounded Radon measures. 16
$\overline{\mathsf{m}}$**:** direction of the variation in the rank-one property at the point $(\bar{t}, \bar{x})$. 71
$\mathcal{M}^+(X)$**:** set of positive Radon measures. 16

$\mathtt{M}$: deformation factor. 44, 51
μ: generic signed Radon measure. 16
μ^β: measure $\mathrm{div}(\beta(\rho)(1,\mathbf{b}))$. 64
$\mu^\beta_{\mathfrak{a}}$: disitegration of the measure $\mathrm{div}(\beta(\rho)(1,\mathbf{b}))$. 63
$|\mu|$: total variation measure of μ. 16
μ^r_x: rescaled μ about $x \in \mathbb{R}^d$. 16
$\mu\llcorner_A$: restriction of the measure μ to the set A. 16
$M(x)$: matrix derivative of the absolutely continuous part of a BV vector field. 20

N: negligible set w.r.t. some measure. 17
$\mathtt{n}$: normal to the rank-one property. 15, 20
$\bar{\mathtt{n}}$: normal to the rank-one property at the point $(\bar t, \bar x)$. 71
$\|\cdot\|$: norm in a generic Banach space. 15
$|\cdot|$: norm in $\mathbb{R}^d$. 15
$\nu^{\mathrm{a.c.}}$: absolutely continuous part of ν. 16
$\nu \ll \mu$: ν is absolute continuous w.r.t. μ. 16
$\nu^\perp$: orthogonal component of ν w.r.t. to another given measure. 16

ω_d: volume of the unit ball in $\mathbb{R}^d$. 15
Ω^ε: perturbation of a proper set constructed in Theorem 4.16. 28
$\mu \perp \nu$: orthogonal measures. 16

$\partial^l Q$: lateral boundary of the set Q. 17
$\partial\Omega^\varepsilon_1$: subset of $\partial\Omega$ defined in Theorem 4.16. 29
$\partial\Omega^\varepsilon_2$: subset of $\partial\Omega$ defined in Theorem 4.16. 29
$\partial^\star F$: reduced boundary of the set of finite perimeter F. 20
$\partial_{x_i} f$: spatial partial derivative along the i-th direction. 15
$\partial_t f_t$: time partial derivative. 15
π: transference plan. 19
ψ: smooth test function. 17
$\mathtt{p}_X$: projection on the space X. 15

Q: sets of particular shape, with some index/apex. 17
$\bar Q$: base of the cylinder $Q_{\ell^\pm_{1,\gamma},\ell}$. 73
Q^ℓ_γ: cylinders of approximate flow. 51
$Q_{\ell^-_{1,\gamma},\ell^+_{1,\gamma},\ell}$: approximate cylnder with shape determined by $\ell^\pm_{1,\gamma}, \ell$. 73

$\mathbb{R}$: real numbers. 15
$\frac{d\nu}{d\mu}$: Radon-Nikodym derivative of ν w.r.t. $\mu \geq 0$. 16
$\mathcal{R}(f)$: range of the function f. 16
$\mathbb{R}^d$: d-dimensional real vector space. 15
$\mathrm{Fr}(A, B)$: relative frontier of A in B. 15
$\mathrm{int}(A, B)$: relative interior of the set A in B. 15
ρ: positive solution to transport equation. 17
ρ^i_Ω: evaluation of the measure η^i_Ω. 34
$\mathring{W}$: trajectories with good intersetion properties in the open graph. 58
$\mathtt{R}^i_\Omega$: i-th restriction operator. 33
ρ^{cr}: $((t,x)$-evaluation of η^{Cr}. 51
$\mathtt{R}_\Omega$: restriction operator. 33
$(\mathtt{R}_\Omega)_\sharp \eta^{\mathrm{out}}$: restrition of $(\mathtt{R}_\Omega)_\sharp \eta$ to the exiting trajectories. 55

S_1: subset of $\partial(\Omega^\varepsilon \setminus \Omega)$ defined in Theorem 4.16. 30
S_2: partition of the set $\partial(\Omega^\varepsilon \setminus \Omega)$, Theorem 4.16. 30
S^-_3: partition of the set $\partial(\Omega^\varepsilon \setminus \Omega)$, Theorem 4.16. 30
S^+_3: partition of the set $\partial(\Omega^\varepsilon \setminus \Omega)$, Theorem 4.16. 30

S_4**:** partition of the set $\partial(\Omega^\varepsilon \setminus \Omega)$, Theorem 4.16. 30
$\mathbb{S}^d$**:** unit sphere of dimension d. 15
$\sigma(f(t))$**:** evaluation of the function f w.r.t. the measure $\rho(t)\mathcal{L}^d$. 44
$o(f)$**:** notation for constant infinitesimal w.r.t. f. 17
$\operatorname{supp} f$**:** support of a function f. 16

$\mathtt{T}$**:** hitting point map. 32
t**:** time coordinate. 15
$t_\gamma^{i,-}$**:** entrance time of γ in Ω. 33
$t_\gamma^{i,+}$**:** exit time of γ in Ω. 33
$\mathtt{T}_\Omega^{i,\pm}$**:** mapping of γ to its Ω entering/exiting point. 33
$\mathsf{Tr}^{\mathrm{in}}(B,\Omega)\cdot\mathbf{n}$**:** distributional inner normal trace. 31
$\mathsf{Tr}^{\mathrm{out}}(B,\Omega)\cdot\mathbf{n}$**:** distributional outer normal trace. 31
$\mathsf{Tr}^{\mathrm{in}}(\mathbf{b},\Omega)$**:** inner trace of the vector fields $\mathbf{b}$. 35
$\mathtt{f}$**:** quotient map for $\{\wp_\mathfrak{a}\}_\mathfrak{a}$. 61

u**:** L^∞-solution to $\operatorname{div}(u\rho(1,\mathbf{b})) \in \mathcal{M}$. 63
$U_\mathtt{x}$**:** neighborhood of $\mathtt{x}$. 15
$\mathcal{U}$**:** function locally approximating $\mathbf{b}$. 72

$\varDelta$**:** set of untangled trajectories. 59
$\varGamma$**:** space of characteristics. 17
$\varGamma^{\mathrm{cr}}$**:** set of trajectories crossing a domain. 51
$\varGamma^{\mathrm{cr}}(\Omega)$**:** set of Ω-crossing trajectories. 54
$\varGamma^{\mathrm{in}}(\Omega)$**:** set of Ω-entering trajectories. 54
φ**:** Convolution kernel. 17
ϖ**:** constant controlling the flux across the lateral boundary of approximate cylinders of flows. 44
ϖ^τ**:** measure controlling the untangling functional. 57, 59
ς_x**:** local representation of a Lipschitz boundary. 17
$\varUpsilon$**:** product space of intervals in $\mathbb{R}$ and curves in $\mathbb{R}^d$. 17
$\varXi$**:** set of uniqueness of η. 53

W**:** set of trajectories with good intersection properties. 45
W_1**:** set of disjoint trajectories. 45
W_2**:** set of trajectories whose intersection is still a trajectory. 45
$w_\mathfrak{a}(t)$**:** density of the disintegration of $\mathcal{L}^{d+1}$ w.r.t. $\{\wp_\mathfrak{a}\}_\mathfrak{a}$. 61
$\wp_\mathfrak{a}$**:** evaluation of the equivalence class $E_\mathfrak{a}$. 60

X**:** generic metric space. 15
$\mathtt{x}$**:** generic point in the metrix space X. 15
x**:** space coordinate. 15
$x_\mathbf{n}$**:** coordinate along $\mathbf{n}$. 15
$x_\mathbf{n}^\perp$**:** coordinates orthogonal to $\mathbf{n}$. 15

ζ_C^τ**:** measure locally controlling the untangling functionals. 57

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S. Bianchini: S.I.S.S.A., via Bonomea 265, 34136 Trieste, Italy
Email address: `bianchin@sissa.it`

P. Bonicatto: S.I.S.S.A., via Bonomea 265, 34136 Trieste, Italy
Email address: `paolo.bonicatto@sissa.it`